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Limiting absorption principle and well-posedness for the Helmholtz equation with sign changing coefficients

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A B S T R A C T

In this paper, we investigate the limiting absorption principle associated to and the well-posedness of the Helmholtz equations with sign changing coefficients which are used to model negative index materials. Using the reflecting technique introduced in [26], we first derive Cauchy problems from these equations. The limiting absorption principle and the well-posedness are then obtained via various a priori estimates for these Cauchy problems. Three approaches are proposed to obtain the a priori estimates. The first one follows from a priori estimates of elliptic systems equipped with complementing boundary conditions due to Agmon, Douglis, and Nirenberg in their classic work [1]. The second approach, which complements the first one, is variational and based on the Dirichlet principle. The last approach, which complements the second one, is also variational and uses the multiplier technique. Using these approaches, we are able to obtain new results on the well-posedness of these equations for which the conditions on the coefficients are imposed "partially" or "not strictly" on the interfaces of sign changing coefficients. This allows us to rediscover and extend known results obtained by the integral method, the pseudo differential operator theory, and the T-coercivity approach. The unique solution, obtained by the limiting absorption principle, is **not** in $H^1_{loc}(\mathbb{R}^d)$ as usual and possibly **not even** in $L^2_{loc}(\mathbb{R}^d)$. The optimality of our results is also discussed.

R É S U M É

Dans cet article, on étudie le principe d'absorption limite et le caractère bien posé des équations de Helmholtz avec changements de signe des coefficients, ce qui modélise des matériaux d'indice négatif. En utilisant la technique de réflexion introduite dans [26], on dérive d'abord des problèmes de Cauchy. Le principe d'absorption limite et le caractère bien posé sont ensuite obtenus grâce à des estimations a priori pour ces problèmes. Trois approches sont proposées pour obtenir ces estimations. La première utilise les estimations a priori des systèmes elliptiques pour des conditions aux limites complémentaires dans l'ouvrage classique [1] d'Agmon, Douglis et Nirenberg. La deuxième approche, qui complète la première, est variationnelle et utilise le principe de Dirichlet. La dernière approche, qui complète la seconde, est également variationnelle et utilise la technique du multiplicateur. Utilisant ces approches, on peut obtenir des nouveaux résultats sur le caractère bien posé de ces équations, pour lesquelles les conditions sur les coefficients sont imposées "partiellement" ou "pas strictement" sur les interfaces où les coefficients

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changent la signe. Cela permet de redécouvrir et d'étendre les résultats connus obtenus par la méthode intégrale, la théorie des opérateurs pseudo differentiels, et l'approche T-coercivité. La solution unique, obtenue par le principe d'absorption limite, **n'est pas** dans $H^1_{loc}(\mathbb{R}^d)$ comme d'habitude et **n'est peut-être même pas** dans $L^2_{loc}(\mathbb{R}^d)$. L'optimalité de nos résultats est également discutée.

1. Introduction

This paper deals with the Helmholtz equation with sign changing coefficients which are used to model negative index materials (NIMs). NIMs were first investigated theoretically by Veselago in [44]. The existence of such materials was confirmed by Shelby, Smith, and Schultz in [42]. The study of NIMs has attracted a lot of attention in the scientific community thanks to their many possible applications such as superlensing and cloaking using complementary media, and cloaking a source via anomalous localized resonance.

We next mention briefly these three applications of NIMs. Superlensing using NIMs was suggested by Veselago in [44] for a slab lens (a slab of index -1) using the ray theory. Later, cylindrical lenses in the two dimensional quasistatic regime, the Veselago slab lens and cylindrical lenses in the finite frequency regime, and spherical lenses in the finite frequency regime were studied by Nicorovici, McPhedran, and Milton in [36], Pendry in [38,39], and Pendry and Ramakrishna in [41] respectively for constant isotropic objects. Superlensing using NIMs (or more precisely using complementary media) for arbitrary objects in the acoustic and electromagnetic settings was established in [27,31] for schemes inspired by [36,39,41] and guided by the concept of reflecting complementary media introduced and studied in [26]. Cloaking using complementary media was suggested and investigated numerically by Lai et al. in [18]. Cloaking an arbitrary inhomogeneous object using complementary media was proved in [30] for the quasi-static regime and later extended in [35] for the finite frequency regime. The schemes used there are inspired by [18] and [26]. Cloaking a source via anomalous localized resonance was discovered by Milton and Nicorovici for constant symmetric plasmonic structures in the two dimensional quasistatic regime in [22] (see also [24,36]) for dipoles. Cloaking an arbitrary source concentrated on a manifold of codimension 1 in an arbitrary medium via anomalous localized resonance was proposed and established in [28,29,33]. Other contributions are [3,4,11,17,34] in which special structures and partial aspects were investigated. A survey on the mathematics progress of these applications can be found in [32]. It is worthy noting that in the applications of NIMs mentioned above, the localized resonance, i.e., the field blows up in some regions and remains bounded in some others as the loss goes to 0, might appear.

In this paper, we investigate the well-posedness of the Helmholtz equation with sign changing coefficients: the stability aspect. To ensure to obtain physical solutions, we also study the limiting absorption principle associated to this equation. Let $k > 0$ and let A be a (real) uniformly elliptic symmetric matrix defined in $\mathbb{R}^d$ ($d \geq 2$), and Σ be a bounded real function defined in $\mathbb{R}^d$ (hence Σ can take both positive and negative values). Assume that

$$A(x) = I \text{ in } \mathbb{R}^d \setminus B_{R_0}, \quad A \text{ is piecewise } C^1,\text{[1]}$$

and

$$\Sigma(x) = 1 \text{ in } \mathbb{R}^d \setminus B_{R_0},$$

for some $R_0 > 0$. Here and in what follows, for $R > 0$, B_R denotes the open ball in $\mathbb{R}^d$ centered at the origin and of radius R. Let $D \subset\subset B_{R_0}$ be a bounded open subset in $\mathbb{R}^d$ of class C^2. Set, for $\delta \geq 0$,

[1] The smoothness assumption of A is only used in the proof of the uniqueness where the unique continuation is applied.

$$s_\delta(x) = \begin{cases} -1 - i\delta & \text{in } D, \\ 1 & \text{in } \mathbb{R}^d \setminus D. \end{cases} \tag{1.1}$$

We are interested in the well-posedness in the class of outgoing solutions of the following equation

$$\operatorname{div}(s_0 A \nabla u_0) + k^2 s_0 \Sigma u_0 = f \text{ in } \mathbb{R}^d, \tag{1.2}$$

and the limiting absorption principle associated with it, i.e., the convergence of u_δ to u_0 (in an appropriate sense) under various conditions on A and Σ as $\delta \to 0_+$. Here $u_\delta \in H^1(\mathbb{R}^d)$ ($\delta > 0$) is the unique solution of the equation

$$\operatorname{div}(s_\delta A \nabla u_\delta) + k^2 s_0 \Sigma u_\delta + i\delta u_\delta = f \text{ in } \mathbb{R}^d. \tag{1.3}$$

Recall that a solution $v \in H^1_{loc}(\mathbb{R}^d \setminus B_R)$ of the equation

$$\Delta v + k^2 v = 0 \text{ in } \mathbb{R}^d \setminus B_R,$$

for some $R > 0$, is said to satisfy the outgoing condition if

$$\partial_r v - ikv = o(r^{-\frac{d-1}{2}}) \text{ as } r = |x| \to +\infty.$$

Physically, k is the frequency, $(s_\delta A, s_0 \Sigma)$ is the material parameter of the medium, and δ describes the loss of the material. We denote

$$\Gamma = \partial D,$$

and, for $\tau > 0$,

$$D_\tau = \big\{x \in D;\ \operatorname{dist}(x, \Gamma) < \tau\big\} \tag{1.4}$$

$$D_{-\tau} = \big\{x \in \mathbb{R}^d \setminus \bar{D};\ \operatorname{dist}(x, \Gamma) < \tau\big\}. \tag{1.5}$$

As usual, $\bar{D}$ denotes the closure of D for a subset D of $\mathbb{R}^d$.

The well-posedness of the Helmholtz equation with sign changing coefficients was first established by Costabel and Stephan in [15]. They proved, by the integral approach, that (1.2) is well-posed if $A = I$ in $\mathbb{R}^d \setminus D$ and $A = \lambda I$ in D provided that λ is a positive constant not equal to 1. Later, Ola in [37] proved, using the integral method and the pseudo-differential operators theory, that (1.2) is well-posed in three and higher dimensions if Γ is strictly convex and connected even though $\lambda = 1$, i.e., $A = I$ in $\mathbb{R}^d$. His result was extended for the case, where Γ has two strictly convex connected components by Kettunen, Lassas, and Ola in [16]. Recently, the well-posedness was extensively studied by Bonnet-Ben Dhia, Ciarlet, and their coauthors in [5–10,13] by T-coercivity approach. This approach was introduced by Bonnet-Ben Dhia, Ciarlet, and Zwölf in [9] and is related to the (Banach–Necas–Babuska) inf-sup condition. The sharpest result for the acoustic setting in this direction, obtained by Bonnet-Ben Dhia, Chesnel, and Ciarlet in [5], is that (1.2) is well-posed in the Fredholm sense in H^1 (this means that the compactness holds[2]), if A is isotropic, i.e., $A = aI$ for some positive function a, and roughly speaking the contrast a_+/a_- is not 1 on Γ.

In this paper, we are interested in the limiting absorbtion principle and the well-posedness of (1.2) for solutions obtained by the limiting absorption process. Our starting point is to obtain Cauchy's problems using the reflecting technique introduced in [26]. To this end, we introduce

[2] They considered the bounded setting and the uniqueness is not ensured in general and is not a consequence of the compactness.

Definition 1. Let $\tau > 0$ and U be a smooth open subset of $\mathbb{R}^d$ such that $\bar{D} \subset U$. A transformation $F : U \setminus \bar{D} \to D_\tau$ is said to be a reflection through Γ if and only if F is a diffeomorphism and $F(x) = x$ on Γ.

Here and in what follows, when we mention a diffeomorphism $F : \Omega \to \Omega'$ for two open subsets Ω, Ω' of $\mathbb{R}^d$, we mean that F is a diffeomorphism, $F \in C^1(\bar{\Omega})$, and $F^{-1} \in C^1(\bar{\Omega}')$.

The idea is simple as follows. Let $F : U \setminus \bar{D} \to D_\tau$ be a reflection through Γ for some smooth open subset U of $\mathbb{R}^d$ such that $\bar{D} \subset U$ and for some $\tau > 0$. Set $v_\delta = u_\delta \circ F^{-1}$. By a change of variables (see Lemma 4), it follows from (1.1) that

$$\begin{gathered}
\operatorname{div}(F_* A \nabla v_\delta) + k^2 F_* \Sigma v_\delta = F_* f + O(\delta v_\delta) \text{ in } D_\tau, \\
\operatorname{div}(A \nabla u_\delta) + k^2 \Sigma u_\delta = s_0^{-1} f + O(\delta u_\delta) + O(\delta f) \text{ in } D_\tau, \\
v_\delta - u_\delta = 0 \text{ on } \Gamma \quad \text{ and } F_* A \nabla v_\delta \cdot \nu - A \nabla u_\delta \big|_D \cdot \nu = i\delta A \nabla u_\delta \big|_D \cdot \nu \text{ on } \Gamma.
\end{gathered}$$

Here and in what follows, for a matrix a, a function σ, and a diffeomorphism T, the following standard notations are used:

$$T_* a(y) = \frac{DT(x) a(x) DT^T(x)}{J(x)} \quad \text{ and } \quad T_* \sigma(y) = \frac{\sigma(x)}{J(x)}, \tag{1.6}$$

where

$$J(x) = |\det DT(x)| \quad \text{ and } \quad x = T^{-1}(y),$$

and on the boundary of a smooth bounded open subset of $\mathbb{R}^d$, ν denotes the normal unit vector directed to its exterior unless otherwise specified. Here $O(v)$ denotes a quantity whose L^2-norm is bounded by $C\|v\|_{L^2}$ for some positive constant C independent of δ and v for $0 < \delta < 1$. We hence obtain Cauchy's problems for (u_δ, v_δ) in D_τ by considering $O(\delta v_\delta)$, $O(\delta u_\delta)$, $O(\delta f)$, and $i\delta A \nabla u_\delta \cdot \nu$ like given data which are formally 0 if $\delta = 0$. The use of reflections to study NIMs was also considered by Milton et al. in [23] and by Bonnet-Ben Dhia, Ciarlet, and their coauthors in their T-coercivity approach. However, there is a difference between the use of reflections in [23], in the T-coercivity approach, and in our work. In [23], the authors used reflections as a change of variables to obtain a new simple setting from an old more complicated one and hence the analysis of the old problem becomes simpler. In the T-coercivity approach, the authors used a standard reflection to construct test functions for the inf-sup condition to obtain an a priori estimate for the solution. Our use of reflections is to derive the Cauchy problems. This can be done in a very flexible way via a change of variables formula stated in Lemma 4 as observed in [26]. The limiting absorption principle and the well-posedness of (1.2) are then based on a priori estimates for these Cauchy problems under various conditions on A, Σ, $F_* A$, and $F_* \Sigma$ in D_τ. Appropriate choices of reflections are important in the applications and discussed later (Corollaries 2, 3, and 4).

In this paper, we introduce three approaches to obtain a priori estimates for the Cauchy problems. The first one follows from a priori estimates for **elliptic systems** imposing **complementing boundary conditions** (see Definition 2) due to Agmon, Douglis, and Nirenberg in their classic work [1]. Applying their result, we can prove in Section 2:

1. Assume that $A_+ := A\big|_{\mathbb{R}^d \setminus D} \in C^1(\bar{D}_{-\tau})$ and $A_- := A\big|_D \in C^1(\bar{D}_\tau)$ for some small positive constant τ, and A_+ and A_- satisfy the (Cauchy) complementing boundary condition on Γ. Then the limiting absorption principle and the well-posedness in $H^1_{loc}(\mathbb{R}^d)$ for (1.2) hold (Theorem 1 in Section 2).

In fact, we establish that the conclusions hold if $F_* A_+$ and A_- satisfy the (Cauchy) complementing boundary condition on Γ where F is the standard reflection in (2.17). Using the characterization of the complementing boundary condition established in Proposition 1, we can prove that F_*A_+ and A_- satisfy the (Cauchy) complementing boundary condition on Γ if and only if A_+ and A_- do; this implies the first result. Using the first result, one obtains new conditions for which the well-posedness and the limiting absorption principle hold. In particular, the condition $A_+ > A_-$ or $A_- > A_+$ on each connected component of Γ is sufficient for the conclusion (see Corollary 1). Here and in what follows, we use the following standard notation for a matrix M: $M > 0$ means that $\langle Mx, x\rangle > 0$ for all $x \neq 0$ where $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^d$. To our knowledge, Corollary 1 is new and cannot be obtained using the known approaches mentioned above. Corollary 1 is in the same spirit of the one of Bonnet-Ben Dhia, Chesnel, and Ciarlet in [5]; nevertheless, A_+ and A_- are not assumed to be isotropic here. Theorem 1 also implies the well-posedness in the case where neither $A_+ > A_-$ nor $A_- > A_+$ holds (see Proposition 1 and Remark 2). One can verify that if $F_*A_+ = A_-$ on Γ then the complementing boundary condition is not satisfied (see Proposition 1). To deal with this situation, we develop a second approach to obtain a priori estimates for the Cauchy problems in Section 3. This approach is variational and based on the Dirichlet principle. Using this approach, we can establish:

2. Assume that there exist $\tau > 0$ (small), a smooth open subset U of $\mathbb{R}^d$ with $\bar{D} \subset U$, and a reflection $F : U \setminus D \to D_\tau$ such that, on every connected component of D_τ,
$$\text{either} \quad A - F_*A \gtrsim \operatorname{dist}(x, \Gamma)^\alpha I \quad \text{or} \quad F_*A - A \gtrsim \operatorname{dist}(x, \Gamma)^\alpha I, \tag{1.7}$$
for some $0 \le \alpha < 2$. Then the limiting absorption principle and the well-posedness for (1.2) hold (Theorem 2 in Section 3).

The unique solution, which is obtained by the limiting absorption principle, might **not** be in $H^1_{loc}(\mathbb{R}^d)$ in this case; the proof of the uniqueness is nonstandard. The appropriate space in which the solution is defined is revealed by the limiting absorption principle; more precisely, by a priori estimates obtained for u_δ defined in (1.3). Once the uniqueness is obtained, the stability is based on a compactness argument. A new compactness criterion in L^2 (Lemma 7) is established in this process and the condition $\alpha < 2$ is required there. Various consequences of this result are given in Section 3 (Corollaries 2 and 3). The choice of the reflections is crucial in deriving these consequences. Theorem 2 implies, unifies, and extends the known results mentioned above. In particular, a variant of the result of Ola in [37] in two dimensions holds and is contained in Theorem 2.

Similar conclusion still holds in the case $F_*A = A$ in D_τ under additional assumptions on Σ and $F_*\Sigma$ in D_τ. To reach the conclusion in this case, we propose a third approach to deal with the Cauchy problems in Section 4. It is variational and based on the multiplier technique, i.e., based on the use of appropriate test functions. In this direction, we can prove the following result:

3. Assume that there exist $\tau > 0$ (small), a smooth open subset U of $\mathbb{R}^d$ with $\bar{D} \subset U$, and a reflection $F : U \setminus D \to D_\tau$ such that either
$$F_*A - A \ge 0 \quad \text{and} \quad \Sigma - F_*\Sigma \gtrsim \operatorname{dist}(x, \Gamma)^\beta \tag{1.8}$$
or
$$A - F_*A \ge 0 \quad \text{and} \quad F_*\Sigma - \Sigma \gtrsim \operatorname{dist}(x, \Gamma)^\beta, \tag{1.9}$$
in each connected component of D_τ for some $\beta > 0$. Then the limiting absorption principle and the well-posedness for (1.2) hold (Theorem 3 in Section 4).

The unique solution, in this case, is **not even** in $L^2_{loc}(\mathbb{R}^d)$ and f is assumed to be 0 near Γ. The appropriate space for which the solution is defined is again revealed by the limiting absorption principle. Once the uniqueness is established, the stability is based on a compactness argument. Due to the lack of L^2-control, the compactness argument used in this case is non-standard and different from the one used in the second setting (see the proofs of Theorem 2 and Theorem 3). A simple application of this result is given in Corollary 4 which is a complement to Corollary 3 in two dimensions. As far as we know, Theorem 3 is the first result on the limiting absorption principle and the well-posedness for the Helmholtz equations with sign changing coefficients where the conditions on the coefficients contain the zero order term Σ.

It is known that in the case $(F_*A, F_*\Sigma) = (A, \Sigma)$ in D_τ, the localized resonance might appear. Media with this property are roughly speaking called reflecting complementary media introduced and studied in [26,31] for the Helmholtz and Maxwell equations respectively. The notion of reflecting complementary media plays an important role in various applications of NIMs mentioned previously as was discussed in [27–31,34,35]. The results obtained in this paper, in particular from the second and the third results, showed that the reflecting complementary property of media is necessary for the occurrence of the resonance. In Section 5, we show that even in the case $(F_*A, F_*\Sigma) = (A, \Sigma)$ in $B(x_0, r_0) \cap D_\tau$ for some $x_0 \in \Gamma$ and $r_0 > 0$, the system is resonant in the following sense (see Proposition 2): There exists f with $\operatorname{supp} f \subset\subset B_{R_0} \setminus \Gamma$ such that $\limsup_{\delta \to 0} \|u_\delta\|_{L^2(K)} = +\infty$ for some $K \subset\subset B_{R_0} \setminus \Gamma$. Here and in what follows $B(x, r)$ denotes the open ball centered at x and of radius r. This result implies the optimality of the second and the third results mentioned above. The proof of Proposition 2 is based on a three sphere inequality and has roots from [29].

The paper is organized as follows. Sections 2, 3, and 4 are devoted to the proof of the three main results mentioned above and their consequences respectively. In Section 5, we discuss the optimality of these results.

2. An approach via a priori estimates of elliptic systems imposed complementing boundary conditions

A useful simple technique suggested to study the Helmholtz equations with sign changing coefficients is the reflecting one introduced in [26]. Applying this technique, we obtain Cauchy problems from the Helmholtz equations with sign changing coefficients. An important part in the investigation of the well-posedness and the limiting absorption principle is then to obtain appropriate a priori estimates for these Cauchy problems. In this section, these follow from an estimate near the boundary of solutions of **elliptic systems** imposed Cauchy data due to Agmon, Douglis, and Nirenberg in their classic work [1] (see also [19]). Before stating the result, let us recall the notation of complementing boundary condition with respect to the Cauchy data derived from [1].

Definition 2. (See Agmon, Douglis, Nirenberg [1].) Two constant positive symmetric matrices A_1 and A_2 are said to satisfy the (Cauchy) complementing boundary condition with respect to direction $e \in \partial B_1$ if and only if for all $\xi \in \mathbb{R}^d_{e,0} \setminus \{0\}$, the only solution $(u_1(x), u_2(x))$ of the form $\big(e^{i\langle y, \xi\rangle} v_1(t), e^{i\langle y, \xi\rangle} v_2(t)\big)$ with $x = y + te$ where $t = \langle x, e\rangle$, of the following system

$$\begin{cases} \operatorname{div}(A_1 \nabla u_1) = \operatorname{div}(A_2 \nabla u_2) = 0 \text{ in } \mathbb{R}^d_{e,+}, \\ u_1 = u_2 \text{ and } A_1 \nabla u_1 \cdot e = A_2 \nabla u_2 \cdot e \text{ on } \mathbb{R}^d_{e,0}, \end{cases}$$

which is bounded in $\mathbb{R}^d_{e,+}$ is $(0, 0)$.

Here and in what follows, for a unit vector $e \in \mathbb{R}^d$, the following notations are used

$$\mathbb{R}^d_{e,+} = \{\xi \in \mathbb{R}^d;\ \langle \xi, e\rangle > 0\} \quad \text{and} \quad \mathbb{R}^d_{e,0} = \{\xi \in \mathbb{R}^d;\ \langle \xi, e\rangle = 0\}. \tag{2.1}$$

Recall that $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^d$.

We are ready to state the main result of this section:

Theorem 1. *Let $f \in L^2(\mathbb{R}^d)$ with $\operatorname{supp} f \subset\subset B_{R_0}$, and let $u_\delta \in H^1(\mathbb{R}^d)$ $(0 < \delta < 1)$ be the unique solution of (1.3). Assume that $A_+ := A\big|_{\mathbb{R}^d \setminus \bar{D}} \in C^1(\bar{D}_{-h})$ and $A_- := A\big|_D \in C^1(\bar{D}_h)$, and $A_+(x), A_-(x)$ satisfy the (Cauchy) complementing boundary condition with respect to $\nu(x)$ for all $x \in \Gamma$. Then*

$$\|u_\delta\|_{H^1(B_R)} \le C_R \|f\|_{L^2(\mathbb{R}^d)} \quad \forall R > 0, \tag{2.2}$$

for some positive constant C_R independent of δ and f. Moreover, (u_δ) converges to u_0 weakly in $H^1_{loc}(\mathbb{R}^d)$ and strongly in $L^2_{loc}(\mathbb{R}^d)$, as $\delta \to 0$, where $u_0 \in H^1_{loc}(\mathbb{R}^d)$ is the unique outgoing solution of (1.2). We also have

$$\|u_0\|_{H^1(B_R)} \le C_R \|f\|_{L^2(\mathbb{R}^d)} \quad \forall R > 0. \tag{2.3}$$

We next give an algebraic characterization of the complementing boundary condition.

Proposition 1. *Let e be a unit vector in $\mathbb{R}^d$ and let A_1 and A_2 be two constant positive symmetric matrices. Then A_1 and A_2 satisfy the (Cauchy) complementing boundary condition with respect to e if and only if*

$$\langle A_2 e, e\rangle \langle A_2 \xi, \xi\rangle - \langle A_2 e, \xi\rangle^2 \neq \langle A_1 e, e\rangle \langle A_1 \xi, \xi\rangle - \langle A_1 e, \xi\rangle^2 \quad \forall \xi \in P \setminus \{0\}, \tag{2.4}$$

where

$$\mathcal{P} := \big\{\xi \in \mathbb{R}^d; \langle \xi, e\rangle = 0\big\}.$$

In particular, if $A_2 > A_1$ then A_1 and A_2 satisfy the (Cauchy) complementing boundary condition with respect to e.

Remark 1. If A_1 and A_2 satisfy the (Cauchy) complementing boundary condition with respect to e then they satisfy the (Cauchy) complementing boundary condition with respect to $-e$.

Remark 2. Assume that A_1 is isotropic, i.e., $A_1 = \lambda I$ for some $\lambda > 0$, and $d = 2$. Then A_1 and A_2 satisfy the complementing boundary condition with respect to e if and only if $\det A_2 \neq \lambda^2$. In general, (2.4) is only required on a subset of $\mathcal{P}$, which is of co-dimension 1.

Using Theorem 1 and Proposition 1, one obtains new conditions for which the well-posedness and the limiting absorption principle hold. In particular, one can immediately derive the following result:

Corollary 1. *Let $f \in L^2(\mathbb{R}^d)$ with $\operatorname{supp} f \subset\subset B_{R_0}$, and let $u_\delta \in H^1(\mathbb{R}^d)$ $(0 < \delta < 1)$ be the unique solution of (1.3). Assume that $A_+ := A\big|_{\mathbb{R}^d \setminus \bar{D}} \in C^1(\bar{D}_{-\tau})$ and $A_- := A\big|_D \in C^1(\bar{D}_\tau)$ for some $\tau > 0$, and $A_+(x) > A_-(x)$ or $A_-(x) > A_+(x)$ for all $x \in \Gamma$. Then*

$$\|u_\delta\|_{H^1(B_R)} \le C_R \|f\|_{L^2(\mathbb{R}^d)} \quad \forall R > 0,$$

for some positive constant C_R independent of δ and f. Moreover, $u_\delta \to u_0$ weakly in $H^1_{loc}(\mathbb{R}^d)$, as $\delta \to 0$, where $u_0 \in H^1_{loc}(\mathbb{R}^d)$ is the unique outgoing solution of (1.2). We also have

$$\|u_0\|_{H^1(B_R)} \le C_R \|f\|_{L^2(\mathbb{R}^d)} \quad \forall R > 0.$$

To our knowledge, Corollary 1 is new and cannot be obtained using the known approaches mentioned in the introduction. Corollary 1 is in the same spirit of the one of Bonnet-Ben Dhia, Chesnel, and Ciarlet in [5]; nevertheless, A_+ and A_- are not assumed to be isotropic here. Using Proposition 1 and applying

Theorem 2, one can also obtain the well-posedness for the case, where neither $A_+ > A_-$ nor $A_- > A_+$ holds (see Remark 2).

The rest of this section contains three subsections. In the first one, we present some lemmas which are used in the proof of Theorem 1. The proof of Theorem 1 is given in the second subsection. In the third subsection, we present the proof of Proposition 1.

2.1. Preliminaries

In this section, we present some lemmas used in the proof of Theorem 1. The first one is on an estimate for solutions to the Helmholtz equation. The proof is based on the unique continuation principle via a compactness argument.

Lemma 1. *Let $d \geq 2$, Ω be a smooth bounded open subset of $\mathbb{R}^d$, $f \in L^2(\Omega)$, and let a be a real uniformly elliptic matrix-valued function and σ be a bounded complex function defined in Ω. Assume that a is piecewise Lipschitz and $v \in H^1(\Omega)$ is a solution to*

$$\operatorname{div}(a\nabla v) + \sigma v = f \text{ in } \Omega.$$

We have

$$\|v\|_{H^1(\Omega)} \leq C\Big(\|f\|_{L^2(\Omega)} + \|v\|_{H^{1/2}(\partial\Omega)} + \|a\nabla v \cdot \nu\|_{H^{-1/2}(\partial\Omega)}\Big), \tag{2.5}$$

for some positive constant C independent of f and v.

Proof. We first establish

$$\|v\|_{L^2(\Omega)} \leq C\Big(\|f\|_{L^2(\Omega)} + \|v\|_{H^{1/2}(\partial\Omega)} + \|a\nabla v \cdot \nu\|_{H^{-1/2}(\partial\Omega)}\Big), \tag{2.6}$$

by contradiction. Here and in what follows in this proof, C denotes a positive constant independent of f, v, and n. Assume that there exist a sequence $(f_n) \subset L^2(\Omega)$ and a sequence $(v_n) \subset H^1(\Omega)$ such that

$$\|v_n\|_{L^2(\Omega)} = 1, \quad \|f_n\|_{L^2(\Omega)} + \|v_n\|_{H^{1/2}(\partial\Omega)} + \|a\nabla v_n \cdot \nu\|_{H^{-1/2}(\partial\Omega)} \leq 1/n \tag{2.7}$$

and

$$\operatorname{div}(a\nabla v_n) + \sigma v_n = f_n \text{ in } \Omega. \tag{2.8}$$

Multiplying the equation of $\bar{v}_n$ (the conjugate of v_n) and integrating on Ω, we obtain

$$\|\nabla v_n\|_{L^2(\Omega)} \leq C\Big(\|v_n\|_{L^2(\Omega)} + \|f_n\|_{L^2(\Omega)} + \|v_n\|_{H^{1/2}(\partial\Omega)} + \|a\nabla v_n \cdot \nu\|_{H^{-1/2}(\partial\Omega)}\Big) \tag{2.9}$$

which implies

$$\|v_n\|_{H^1(\Omega)} \leq C.$$

Without loss of generality, one might assume that (v_n) converges to v weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$. It follows from (2.7) and (2.8) that

$$\operatorname{div}(a\nabla v) + \sigma v = 0 \text{ in } \Omega$$

and $v = A\nabla v \cdot \nu = 0$ on $\partial\Omega$. By the unique continuation principle, see e.g., [40], $v = 0$ in Ω. This contradicts the fact, by (2.7),

$$\|v\|_{L^2(\Omega)} = 1.$$

Hence (2.6) holds. The conclusion now follows from (2.9) where v_n is replaced by v. □

Remark 3. Assume that $a \in C^1(\bar{\Omega})$. Using a three spheres inequality, see e.g., [2,35], one can choose the constant C depending only on Ω, the elliptic and Lipschitz constants of a, the boundedness of a and σ.

The following lemma is used to obtain an a priori estimate for u_δ defined in (1.3).

Lemma 2. *Let $f \in L^2(\mathbb{R}^d)$ with $\operatorname{supp} f \subset\subset B_{R_0}$ and let $u_\delta \in H^1(\mathbb{R}^d)$ be the unique solution of (1.3). Then*

$$\|u_\delta\|^2_{H^1(\mathbb{R}^d)} \le C\Big(\frac{1}{\delta}\Big|\int_{\mathbb{R}^d} f\bar{u}_\delta\Big| + \|f\|^2_{L^2(\mathbb{R}^d)}\Big), \tag{2.10}$$

for some positive constant C independent of f and δ. Consequently,

$$\|u_\delta\|_{H^1(\mathbb{R}^d)} \le \frac{C}{\delta}\|f\|_{L^2(\mathbb{R}^d)}.$$

Proof. Multiplying the equation of u_δ by $\bar{u}_\delta$ and integrating on $\mathbb{R}^d$, we have

$$-\int_{\mathbb{R}^d} \langle s_\delta A\nabla u_\delta, \nabla u_\delta\rangle + \int_{\mathbb{R}^d} k^2 s_0 \Sigma |u_\delta|^2 + i\delta|u_\delta|^2 = \int_{\mathbb{R}^d} f\bar{u}_\delta. \tag{2.11}$$

Considering the imaginary part of (2.11), we obtain

$$\int_D |\nabla u_\delta|^2 + \int_{\mathbb{R}^d} |u_\delta|^2 \le \frac{C}{\delta}\Big|\int_{\mathbb{R}^d} f\bar{u}_\delta\Big|.$$

This implies

$$\|u_\delta\|^2_{H^{1/2}(\partial D)} + \|A\nabla u_\delta \cdot \nu\|^2_{H^{-1/2}(\partial D)} \le \frac{C}{\delta}\Big|\int_{\mathbb{R}^d} f\bar{u}_\delta\Big| + C\|f\|^2_{L^2(\mathbb{R}^d)}.$$

Let Ω be the complement of the unbounded connected component of $\mathbb{R}^d \setminus D$. Applying Lemma 1, we have

$$\|u_\delta\|^2_{H^1(\Omega)} \le \frac{C}{\delta}\Big|\int_{\mathbb{R}^d} f\bar{u}_\delta\Big| + C\|f\|^2_{L^2(\Omega)}. \tag{2.12}$$

Considering the real part of (2.11) and using (2.12), we obtain

$$\|u_\delta\|^2_{H^1(\mathbb{R}^d)} \le C\Big(\frac{1}{\delta}\Big|\int_{\mathbb{R}^d} f\bar{u}_\delta\Big| + \|f\|^2_{L^2(\mathbb{R}^d)}\Big).$$

The proof is complete. □

The following lemma on the stability of the outgoing solution is standard (see, e.g., [20][3]).

Lemma 3. *Let $\Omega \subset B_{R_0}$ be a smooth open subset of $\mathbb{R}^d$, and let $f \in L^2(\mathbb{R}^d \setminus \Omega)$ and $g \in H^{\frac{1}{2}}(\partial\Omega)$. Assume that $\mathbb{R}^d \setminus \Omega$ is connected,* $\operatorname{supp} f \subset B_{R_0}$*, and $v \in H^1_{loc}(\mathbb{R}^d)$ is the unique outgoing solution of*

$$\begin{cases} \Delta v + k^2 v = f & \text{in } \mathbb{R}^d \setminus \Omega, \\ v = g & \text{on } \partial\Omega. \end{cases}$$

Then

$$\|v\|_{H^1(B_r \setminus \Omega)} \leq C_r \big(\|f\|_{L^2(\mathbb{R}^d \setminus \Omega)} + \|g\|_{H^{\frac{1}{2}}(\partial\Omega)}\big) \quad \forall r > 0,$$

for some positive constants $C_r = C(r, k, \Omega, R_0, d)$.

We next recall the following result [26, Lemma 2], a change of variables formula, which is used repeatedly in this paper.

Lemma 4. *Let $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_3$ be three smooth bounded open subsets of $\mathbb{R}^d$. Let $a \in [L^\infty(\Omega_2 \setminus \Omega_1)]^{d\times d}$, $\sigma \in L^\infty(\Omega_2 \setminus \Omega_1)$ and let T be a diffeomorphism from $\Omega_2 \setminus \bar{\Omega}_1$ onto $\Omega_3 \setminus \bar{\Omega}_2$ such that $T(x) = x$ on $\partial\Omega_2$. Assume that $u \in H^1(\Omega_2 \setminus \Omega_1)$ and set $v = u \circ T^{-1}$. Then*

$$\operatorname{div}(a\nabla u) + \sigma u = f \text{ in } \Omega_2 \setminus \Omega_1,$$

for some $f \in L^2(\Omega_2 \setminus \Omega_1)$, if and only if

$$\operatorname{div}(T_* a\nabla v) + T_*\sigma v = T_* f \text{ in } \Omega_3 \setminus \Omega_2. \tag{2.13}$$

Moreover,

$$v = u \quad \text{and} \quad T_* a\nabla v \cdot \nu = -a\nabla u \cdot \nu \text{ on } \partial\Omega_2. \tag{2.14}$$

Recall that $T_* a$, $T_*\sigma$, and $T_* f$ are given in (1.6).

2.2. Proof of Theorem 1

We first establish the uniqueness for (1.2). Assume that $f = 0$. We prove that $u_0 = 0$ if $u_0 \in H^1_{loc}(\mathbb{R}^d)$ is an outgoing solution of (1.2). The proof is quite standard as in the usual case, in which the coefficients are positive. Multiplying the equation by $\bar{u}_0$, integrating on B_R, and considering the imaginary part, we have, by letting $R \to +\infty$,

$$\lim_{R\to+\infty} \int_{\partial B_R} |u_0|^2 = 0.$$

Here the outgoing condition is used. By Rellich's lemma (see, e.g., [14]), $u_0 = 0$ in $\mathbb{R}^d \setminus B_{R_0}$. It follows from the unique continuation principle that $u_0 = 0$. The uniqueness is proved.

[3] In [20], the proof is given only for $d = 2, 3$. However, the proof in the case $d > 3$ is similar to the case $d = 3$.

We next establish (2.2). Applying Lemma 2, we have

$$\|u_\delta\|_{H^1(\mathbb{R}^d)} \le \frac{C}{\delta}\|f\|_{L^2(\mathbb{R}^d)}. \tag{2.15}$$

In this proof, C denotes a positive constant independent of δ and f. Using the difference method due to Nirenberg (see, e.g., [12]), one has[4]

$$u_\delta \in H^2(D_{-\tau} \cup D_\tau). \tag{2.16}$$

For $\tau > 0$ small, define $F : D_{-\tau} \to D_\tau$ as follows

$$F(x_\Gamma + t\nu(x_\Gamma)) = x_\Gamma - t\nu(x_\Gamma) \quad \forall\, x_\Gamma \in \Gamma,\, t \in (-\tau, 0). \tag{2.17}$$

Let v_δ be the reflection of u_δ through Γ by F, i.e.,

$$v_\delta = u_\delta \circ F^{-1} \text{ in } D_\tau.$$

By Lemma 4, we have

$$\operatorname{div}(F_* A \nabla v_\delta) + k^2 F_*\Sigma v_\delta + i\delta F_* 1 v_\delta = F_* f \text{ in } D_\tau,$$

and

$$v_\delta - u_\delta\big|_D = 0, \quad F_* A\nabla v_\delta \cdot \nu - A\nabla u_\delta\big|_D \cdot \nu = i\delta A \nabla u_\delta\big|_D \cdot \nu \text{ on } \Gamma.$$

Recall that

$$\operatorname{div}(A\nabla u_\delta) + k^2 \Sigma u_\delta + k^2(s_\delta^{-1}s_0 - 1)\Sigma u_\delta + i\delta s_\delta^{-1} u_\delta = s_\delta^{-1} f \text{ in } D_\tau.$$

Note that A_+ and A_- satisfy the complementing boundary condition on Γ if and only if F_*A_+ and A_- satisfy the complementing boundary condition on Γ by (2.4) in Proposition 1. Applying the result of Agmon, Douglis, and Nirenberg [1, Theorem 10.2], we have

$$\begin{aligned} &\|u_\delta\|_{H^2(D_{\tau/2})} + \|v_\delta\|_{H^2(D_{\tau/2})} \\ &\quad \le C\Big(\|u_\delta\|_{H^1(D_\tau)} + \|v_\delta\|_{H^1(D_\tau)} + \|i\delta A\nabla u_\delta\big|_D \cdot \nu\|_{H^{1/2}(\Gamma)} + \|f\|_{L^2(\mathbb{R}^d)}\Big). \end{aligned} \tag{2.18}$$

Since

$$\|A\nabla u_\delta\big|_D \cdot \nu\|_{H^{1/2}(\Gamma)} \le C\Big(\|u_\delta\|_{H^2(D_{\tau/2})} + \|f\|_{L^2(\mathbb{R}^d)}\Big),$$

it follows that, for small δ,

$$\|u_\delta\|_{H^2(D_{\tau/2})} + \|v_\delta\|_{H^2(D_{\tau/2})} \le C\Big(\|u_\delta\|_{H^1(D_\tau)} + \|v_\delta\|_{H^1(D_\tau)} + \|f\|_{L^2(\mathbb{R}^d)}\Big). \tag{2.19}$$

Using the inequality, for $\lambda > 0$,

$$\|\varphi\|_{H^1(D_{\tau/2})} \le \lambda\|\varphi\|_{H^2(D_{\tau/2})} + C_\lambda\|\varphi\|_{L^2(D_{\tau/2})},$$

[4] We do not claim that $u \in H^2_{loc}(\mathbb{R}^d)$; this fact is not true in general.

we derive from Lemmas 1 and 3 that, for small δ,

$$\|u_\delta\|_{H^2(D_{\tau/2})} + \|v_\delta\|_{H^2(D_{\tau/2})} + \|u_\delta\|_{H^1(B_R)} \le C_R\Big(\|u_\delta\|_{L^2(D_{-\tau}\cup D_\tau)} + \|f\|_{L^2(\mathbb{R}^d)}\Big) \quad \forall\, R>0. \tag{2.20}$$

The proof now follows by a standard compactness argument. We first claim that

$$\|u_\delta\|_{L^2(B_{R_0})} \le C\|f\|_{L^2(\mathbb{R}^d)}. \tag{2.21}$$

Indeed, assume that this is not true. By (2.15), there exist a sequence $(\delta_n) \to 0_+$ and a sequence (f_n) such that $\operatorname{supp} f_n \subset B_{R_0}$,

$$\|u_{\delta_n}\|_{L^2(B_{R_0})} = 1, \quad \text{and} \quad \|f_n\|_{L^2(\mathbb{R}^d)} \to 0.$$

We derive from (2.20) that (u_{δ_n}) is bounded in $H^1_{loc}(\mathbb{R}^d)$. Without loss of generality, one might assume that (u_{δ_n}) converges to u_0 weakly in $H^1_{loc}(\mathbb{R}^d)$ and strongly in $L^2_{loc}(\mathbb{R}^d)$. Then $u_0 \in H^1_{loc}(\mathbb{R}^d)$,

$$\operatorname{div}(s_0 A\nabla u_0) + k^2 s_0 \Sigma u_0 = 0 \text{ in } \mathbb{R}^d,$$

and u_0 satisfies the outgoing condition by the limiting absorption principle. It follows that $u_0 = 0$ in $\mathbb{R}^d$ by the uniqueness. This contradicts the fact $\|u_0\|_{L^2(B_{R_0})} = \lim_{n\to+\infty}\|u_{\delta_n}\|_{L^2(B_{R_0})} = 1$. Hence (2.21) holds.

A combination of (2.15), (2.20), and (2.21) yields

$$\|u_\delta\|_{H^1(B_R)} \le C_R\|f\|_{L^2(\mathbb{R}^d)}. \tag{2.22}$$

Hence for any sequence $(\delta_n) \to 0$, there exists a subsequence (δ_{n_k}) such that $(u_{\delta_{n_k}})$ converges to u_0 weakly in $H^1_{loc}(\mathbb{R}^d)$ and strongly in $L^2_{loc}(\mathbb{R}^d)$. Moreover, $u_0 \in H^1_{loc}(\mathbb{R}^d)$,

$$\operatorname{div}(s_0 A\nabla u_0) + k^2 s_0 \Sigma u_0 = f \text{ in } \mathbb{R}^d,$$

and u_0 satisfies the outgoing condition. Since the limit u_0 is unique, (u_δ) converges to u_0 weakly in $H^1_{loc}(\mathbb{R}^d)$ and strongly in $L^2(\mathbb{R}^d)$ as $\delta \to 0$. The proof is complete. □

2.3. Proof of Proposition 1

Using a rotation if necessary, without lost of generality, one may assume that $e = e_d := (0, \cdots, 0, 1)$. Denote $x = (x', t) \in \mathbb{R}^{d-1} \times \mathbb{R}$. Fix a non-zero vector $\xi' = (\xi_1, \cdots, \xi_{d-1}) \in \mathbb{R}^{d-1}$ and denote $\xi = (\xi', 0)$. Since $u_j(x) = e^{i\langle x, \xi\rangle} v_j(t)$ $(j = 1, 2)$ is a solution to the equation

$$\operatorname{div}(A_j \nabla u_j) = 0 \text{ in } \mathbb{R}^{d-1} \times (0, +\infty),$$

it follows that, for $j = 1, 2$,

$$a_j v_j''(t) + 2ib_j v_j'(t) - c_j v_j(t) = 0 \text{ for } t > 0,$$

where

$$a_j = (A_j)_{d,d}, \quad b_j = \sum_{k=1}^{d-1}(A_j)_{d,k}\xi_k, \quad \text{and} \quad c_j = \sum_{k=1}^{d-1}\sum_{l=1}^{d-1}(A_j)_{k,l}\xi_k\xi_l.$$

Here $(A_j)_{k,l}$ denotes the (k,l) component of A_j for $j=1,2$ and the symmetry of A_j is used. Define, for $j=1,2$,

$$\Delta_j = -b_j^2 + a_j c_j.$$

Since A_j is symmetric and positive, it is clear that, for $j=1,2$,

$$a_j = \langle A_j e_d, e_d\rangle > 0, \quad b_j = \langle A_j \xi, e_d\rangle, \quad \text{and} \quad \Delta_j = \langle A_j e_d, e_d\rangle \langle A_j \xi, \xi\rangle - \langle A_j e_d, \xi\rangle^2 > 0.$$

Since v_j is required to be bounded, we have

$$v_j(t) = \alpha_j e^{\eta_j t},$$

for some $\alpha_j \in \mathbb{C}$, where

$$\eta_j = (-ib_j - \sqrt{\Delta_j})/a_j.$$

Using the fact that $u_1 = u_2$ and $A_1 \nabla u_1 \cdot e_d = A_2 \nabla u_2 \cdot e_d$, we have

$$\alpha_1 = \alpha_2 \quad \text{and} \quad \alpha_1 \Big(\langle i A_2 \xi + \eta_2 A_2 e_d, e_d\rangle - \langle i A_1 \xi + \eta_1 A_1 e_d, e_d\rangle \Big) = 0.$$

The complementing boundary condition is now equivalent to the fact that

$$\Delta_2 \neq \Delta_1,$$

for all non-zero $\xi = (\xi', 0) \in \mathbb{R}^d$. Condition (2.4) is proved.

It remains to prove that if $A_2 > A_1$ then (2.4) holds. Define $M = A_2 - A_1$, fix $\xi \in \mathcal{P} \setminus \{0\}$, and set

$$\Delta = \langle A_2 e, e\rangle \langle A_2 \xi, \xi\rangle - \langle A_2 e, \xi\rangle^2 - \Big(\langle A_1 e, e\rangle \langle A_1 \xi, \xi\rangle - \langle A_1 e, \xi\rangle^2 \Big).$$

Using the fact $A_2 = A_1 + M$, after a straightforward computation, we obtain

$$\Delta = \langle Me, e\rangle \langle A_1 \xi, \xi\rangle + \langle M\xi, \xi\rangle \langle A_1 e, e\rangle + \langle Me, e\rangle \langle M\xi, \xi\rangle - 2\langle Me, \xi\rangle \langle A_1 e, \xi\rangle - \langle Me, \xi\rangle^2. \tag{2.23}$$

We have, by Cauchy's inequality,

$$\langle Me, e\rangle \langle A_1 \xi, \xi\rangle + \langle M\xi, \xi\rangle \langle A_1 e, e\rangle \geq 2\Big(\langle M\xi, \xi\rangle \langle Me, e\rangle \langle A_1 e, e\rangle \langle A_1 \xi, \xi\rangle \Big)^{1/2}. \tag{2.24}$$

Since M and A_1 are symmetric and positive and $\langle \xi, e\rangle = 0$, we obtain, by Cauchy–Schwarz's inequality,

$$\langle Me, e\rangle \langle M\xi, \xi\rangle \langle A_1 e, e\rangle \langle A_1 \xi, \xi\rangle > \langle Me, \xi\rangle^2 \langle A_1 e, \xi\rangle^2 \tag{2.25}$$

and

$$\langle Me, e\rangle \langle M\xi, \xi\rangle > \langle M\xi, e\rangle^2. \tag{2.26}$$

A combination of (2.23), (2.24), (2.25), and (2.26) yields

$$\Delta > 0.$$

The proof is complete. $\square$

3. A variational approach via the Dirichlet principle

In this section, we develop a variational method, which complements the one in Section 2, to deal with a class of A in which F_*A_+ might be equal to A_- on Γ and A_+ and A_- are not supposed to be smooth near Γ; this is not covered by Theorem 1. One motivation comes from the work of Ola in [37]. The other is from the work of Bonnet-Ben Dhia, Chesnel, and Ciarlet in [5], where the smoothness of the coefficients is not required.

The following result is the main result of this section.

Theorem 2. *Let $f \in L^2(\mathbb{R}^d)$ with $\operatorname{supp} f \subset B_{R_0}$, and let $u_\delta \in H^1(\mathbb{R}^d)$ $(0 < \delta < 1)$ be the unique solution of (1.3). Assume that there exists a reflection F from $U \setminus D$ onto D_τ for some $\tau > 0$ and for some smooth open subset U of $\mathbb{R}^d$ with $\bar{D} \subset U$ such that*

$$\text{either} \quad A - F_*A \geq c\operatorname{dist}(x,\Gamma)^\alpha I \quad \text{or} \quad F_*A - A \geq c\operatorname{dist}(x,\Gamma)^\alpha I, \tag{3.1}$$

on each connected component of D_τ, for some $c > 0$, and for some $0 < \alpha < 2$. Set $v_\delta = u_\delta \circ F^{-1}$ in D_τ. Then

$$\|u_\delta\|_{L^2(B_R)} + \|u_\delta - v_\delta\|_{H^1(D_\tau)} + \Big(\int_{D_\tau} \big|\langle (A - F_*A)\nabla u_\delta, \nabla u_\delta\rangle\big|\Big)^{1/2} \leq C_R\|f\|_{L^2(\mathbb{R}^d)}. \tag{3.2}$$

Moreover, (u_δ) converges to u_0 weakly in $H^1_{loc}(\mathbb{R}^d \setminus \Gamma)$ and strongly in $L^2_{loc}(\mathbb{R}^d)$ as $\delta \to 0$, where $u_0 \in H^1_{loc}(\mathbb{R}^d \setminus \Gamma) \cap L^2_{loc}(\mathbb{R}^d)$ is the **unique** *outgoing solution of (1.2) such that the LHS of (3.3) is finite, where $v_0 := u_0 \circ F^{-1}$ in D_τ. Consequently,*

$$\|u_0\|_{L^2(B_R)} + \|u_0 - v_0\|_{H^1(D_\tau)} + \Big(\int_{D_\tau} \big|\langle (A - F_*A)\nabla u_0, \nabla u_0\rangle\big|\Big)^{1/2} \leq C_R\|f\|_{L^2(\mathbb{R}^d)}. \tag{3.3}$$

Here C_R denotes a positive constant independent of f and δ.

Remark 4. We only make the assumption on the lower bound of $F_*A - A$ or $A - F_*A$ in (3.1), not on the upper bound.

The solution u_0 in Theorem 2 is not in $H^1_{loc}(\mathbb{R}^d)$ as usual. The meaning of the solution is given in the following definition:

Definition 3. Let $f \in L^2(\mathbb{R}^d)$ with compact support and let F be a reflection from $U \setminus D$ to D_τ for some $\tau > 0$ (small) and for some smooth open set U with $\bar{D} \subset U$ such that (3.1) holds. A function $u_0 \in H^1_{loc}(\mathbb{R}^d \setminus \Gamma) \cap L^2_{loc}(\mathbb{R}^d)$ such that the LHS of (3.3) is finite is said to be a solution of (1.2) if

$$\operatorname{div}(s_0 A\nabla u_0) + k^2 s_0 \Sigma u_0 = f \text{ in } \mathbb{R}^d \setminus \Gamma, \tag{3.4}$$

$$u_0\big|_D - v_0 = 0 \quad \text{and} \quad (F_*A\nabla v_0 - A\nabla u_0\big|_D) \cdot \nu = 0 \text{ on } \Gamma, \tag{3.5}$$

and

$$\lim_{t \to 0_+} \int_{\partial D_t \setminus \Gamma} \big(F_*A\nabla v_0 \cdot \nu \bar{v}_0 - A\nabla u_0 \cdot \nu \bar{u}_0\big) = 0. \tag{3.6}$$

Remark 5. Since $u_0 - v_0 \in H^1(D_\tau)$ and $\operatorname{div}(F_* A\nabla v_0 - A\nabla u_0) \in L^2(D_\tau)$ (the LHS of (3.3) is finite), it follows that $u_0\big|_D - v_0 \in H^{1/2}(\Gamma)$ and $(F_* A\nabla v_0 - A\nabla u_0\big|_D) \cdot \nu \in H^{-1/2}(\Gamma)$. Hence requirement (3.5) makes sense. It is clear that the definition of weak solutions in Definition 3 coincides with the standard definition of weak solutions when $\alpha = 0$ by Lemma 4. Requirements in (3.5) can be seen as generalized transmission conditions.

The proof of Theorem 2 is based on the Dirichlet principle. The key observation is that the Cauchy data provides the energy of a solution to an elliptic equation (Lemma 5). The proof is also based on a new compactness criterion in L^2 (Lemma 7). The requirement $\alpha < 2$ is used in the compactness argument; we do not know if this condition is necessary. As a direct consequence of Theorem 2 with $\alpha = 0$, we obtain the following result:

Corollary 2. *Let $f \in L^2(\mathbb{R}^d)$ with $\operatorname{supp} f \subset B_{R_0}$, and let $u_\delta \in H^1(\mathbb{R}^d)$ $(0 < \delta < 1)$ be the unique solution of (1.3). Assume that $A \circ F^{-1}(x)$* **or** *$A(x)$ is* **isotropic** *for every $x \in D_\tau$, and*

$$\textit{either} \quad A \circ F^{-1}(x) - A(x) \geq cI \quad \textit{or} \quad A(x) - A \circ F^{-1}(x) \geq cI \tag{3.7}$$

in each connected component D_τ for some small $\tau > 0$ and for some $c > 0$, where $F\big(x_\Gamma + t\nu(x_\Gamma)\big) := x_\Gamma - t\nu(x_\Gamma)$ for $x_\Gamma \in \Gamma$ and $t \in (-\tau, \tau)$. Then

$$\|u_\delta\|_{H^1(B_R)} \leq C_R \|f\|_{L^2(\mathbb{R}^d)}.$$

Moreover, $u_\delta \to u_0$ weakly in $H^1_{loc}(\mathbb{R}^d)$ as $\delta \to 0$, where $u_0 \in H^1_{loc}(\mathbb{R}^d)$ is the unique outgoing solution of (1.2) and

$$\|u_0\|_{H^1(B_R)} \leq C_R \|f\|_{L^2(\mathbb{R}^d)}.$$

Remark 6. Applying Corollary 2, one rediscovers and extends the result obtained by Bonnet-Ben Dhia, Chesnel, and Ciarlet in [5] where A_+ and A_- are both assumed to be isotropic.

We next present another consequence of Theorem 2 for the case $\alpha = 1$. The following notation is used.

Definition 4. The boundary Γ of D is called strictly convex if all its connected components are the boundary of strictly convex sets.

We are ready to present

Corollary 3. *Let $d \geq 3$, $f \in L^2(\mathbb{R}^d)$ with $\operatorname{supp} f \subset B_{R_0}$, and let $u_\delta \in H^1(\mathbb{R}^d)$ $(0 < \delta < 1)$ be the unique solution of (1.3). Assume that D is of class C^3, A is* **isotropic** *and* **constant** *in the orthogonal direction of Γ in a neighborhood of Γ, i.e., $A(x_\Gamma + t\nu(x_\Gamma))$ is independent of $t \in (-\tau_0, \tau_0)$ for $x_\Gamma \in \Gamma$ and for some small positive constant τ_0, and Γ is* **strictly convex**. *There exist $c > 0$, $\tau > 0$, a smooth open set $U \supset\supset D$, a reflection $F : U \setminus D \to D_\tau$ such that $F_* A - A \geq c \operatorname{dist}(x, \Gamma) I$ or $A - F_* A \geq c \operatorname{dist}(x, \Gamma) I$ on each connected component of D_τ. As a consequence, u_δ satisfies (3.2) with $\alpha = 1$. Moreover, $u_\delta \to u_0$ weakly in $H^1_{loc}(\mathbb{R}^d \setminus \Gamma)$ as $\delta \to 0$, where $u_0 \in H^1_{loc}(\mathbb{R}^d \setminus \Gamma) \cap L^2_{loc}(\mathbb{R}^d)$ is the unique outgoing solution of (1.2) and u_0 satisfies (3.3).*

Remark 7. In particular, if A is **isotropic** and **constant** in each connected component of a neighborhood of Γ, then the conclusion of Corollary 3 holds.

Remark 8. Applying Corollary 2, one rediscovers and extends the well-posedness result obtained by Ola [37] and Kettunen, Lassas, and Ola in [16] where $A = I$ in D and Γ has one or two connected components.

Remark 9. Corollary 3 does **not** hold in two dimensions. The strict convexity of Γ is necessary in three dimensions. In four or higher dimensions, the strict convexity of Γ can be relaxed (see Remark 11).

The rest of this section containing three subsections is organized as follows. In the first subsection, we present some lemmas used in the proof of Theorem 2. The second and the third subsections are devoted to the proof of Theorem 2 and Corollary 3 respectively.

3.1. Some useful lemmas

We begin with the following lemma which plays an important role in the proof of Theorem 2.

Lemma 5. *Let Ω be a smooth bounded open subset of $\mathbb{R}^d$, and A_1 and A_2 be two symmetric uniformly elliptic matrices defined in Ω. Let $f_1, f_2 \in L^2(\Omega)$, $h \in H^{-1/2}(\partial\Omega)$ and let $u_1, u_2 \in H^1(\Omega)$ be such that*

$$-\operatorname{div}(A_1 \nabla u_1) = f_1 \quad \text{and} \quad -\operatorname{div}(A_2 \nabla u_2) = f_2 \text{ in } \Omega, \tag{3.8}$$

$$u_1 = u_2 \quad \text{and} \quad A_1 \nabla u_1 \cdot \nu = A_2 \nabla u_2 \cdot \nu + h \text{ on } \partial\Omega. \tag{3.9}$$

Assume that

$$A_1 \geq A_2 \text{ in } \Omega. \tag{3.10}$$

Then

$$\int_\Omega \langle (A_1 - A_2)\nabla u_1, \nabla u_1 \rangle + \int_\Omega |\nabla(u_1 - u_2)|^2 \leq C\Big(\|(f_1, f_2, u_1, u_2)\|^2_{L^2(\Omega)} + \|h\|_{H^{-1/2}(\partial\Omega)} \|(u_1, u_2)\|_{H^{1/2}(\partial\Omega)} \Big). \tag{3.11}$$

Proof. By considering the real part and the imaginary part separately, without loss of generality, one may assume that all functions in Lemma 5 are real. Set

$$\mathcal{M} = \|(f_1, f_2, u_1, u_2)\|^2_{L^2(\Omega)} + \|h\|_{H^{-1/2}(\partial\Omega)} \|(u_1, u_2)\|_{H^{1/2}(\partial\Omega)}.$$

Multiplying the equation of u_j by u_j (for $j = 1, 2$) and integrating on Ω, we have

$$\int_\Omega \langle A_j \nabla u_j, \nabla u_j \rangle = \int_\Omega f_j u_j + \int_{\partial\Omega} A_j \nabla u_j \cdot \nu \, u_j. \tag{3.12}$$

Using (3.8) and (3.9), we derive from (3.12) that

$$\int_\Omega \langle A_1 \nabla u_1, \nabla u_1 \rangle - \langle A_2 \nabla u_2, \nabla u_2 \rangle \leq C\mathcal{M}. \tag{3.13}$$

Here and in what follows, C denotes a positive constant independent of f_j, h, u_j for $j = 1, 2$. By the Dirichlet principle, we have

$$\frac{1}{2}\int_{\Omega}\langle A_2\nabla u_2, \nabla u_2\rangle - \int_{\Omega} f_2 u_2 - \int_{\partial\Omega} A_2\nabla u_2\cdot\nu\, u_2$$
$$\leq \frac{1}{2}\int_{\Omega}\langle A_2\nabla u_1, \nabla u_1\rangle - \int_{\Omega} f_2 u_1 - \int_{\partial\Omega} A_2\nabla u_2\cdot\nu\, u_1. \tag{3.14}$$

A combination of (3.8), (3.9), and (3.14) yields

$$\int_{\Omega}\langle A_2\nabla u_2, \nabla u_2\rangle - \langle A_2\nabla u_1, \nabla u_1\rangle \leq C\mathcal{M}. \tag{3.15}$$

Adding (3.13) and (3.15), we obtain

$$\int_{\Omega}\langle (A_1 - A_2)\nabla u_1, \nabla u_1\rangle \leq C\mathcal{M}. \tag{3.16}$$

Set

$$w = u_1 - u_2 \text{ in } \Omega.$$

We have, in Ω,

$$\begin{aligned}\operatorname{div}(A_2\nabla w) = \operatorname{div}(A_2\nabla u_1) - \operatorname{div}(A_2\nabla u_2) &= \operatorname{div}(A_1\nabla u_1) - \operatorname{div}(A_2\nabla u_2) + \operatorname{div}([A_2 - A_1]\nabla u_1)\\ &= -f_1 + f_2 + \operatorname{div}([A_2 - A_1]\nabla u_1).\end{aligned}$$

Multiplying this equation by w, integrating on Ω, we obtain, by (3.8) and (3.9),

$$\int_{\Omega}|\nabla w|^2 \leq \int_{\Omega} C|\langle (A_1 - A_2)\nabla u_1, \nabla w\rangle| + C\mathcal{M}. \tag{3.17}$$

Since $A_1 > A_2$ and A_1 and A_2 are symmetric, we have, for any $\lambda > 0$,

$$\int_{\Omega}|\langle (A_1 - A_2)\nabla u_1, \nabla w\rangle| \leq \lambda\int_{\Omega}|\langle (A_1 - A_2)\nabla u_1, \nabla u_1\rangle| + \frac{1}{4\lambda}\int_{\Omega}|\langle (A_1 - A_2)\nabla w, \nabla w\rangle|.$$

It follows from (3.16) and (3.17) that

$$\int_{\Omega}|\nabla w|^2 \leq C\mathcal{M}. \tag{3.18}$$

The conclusion now follows from (3.16) and (3.18). The proof is complete. □

We next recall Hardy's inequalities (see, e.g., [21]).

Lemma 6. *Let Ω be a smooth bounded open subset of $\mathbb{R}^d$. Then, for all $u \in H^1(\Omega)$, and for $\alpha > 1$,*

$$\int_{\Omega}\operatorname{dist}(x, \partial\Omega)^{\alpha-2}|u(x)|^2\, dx \leq C_{\alpha,\Omega}\int_{\Omega}\Big(\operatorname{dist}(x, \partial\Omega)^{\alpha}|\nabla u(x)|^2 + |u(x)|^2\Big)\, dx. \tag{3.19}$$

Here $C_{\alpha,\Omega}$ is a positive constant independent of u.

Remark 10. Lemma 6 also holds for Lipschitz domains, see [25, Theorem 1.5].

Using Lemma 6, we can prove the following compactness result which is used in the compactness argument in the proof of Theorem 2.

Lemma 7. *Let $0 \le \alpha < 2$, Ω be a smooth bounded open subset of $\mathbb{R}^d$, and $(u_n) \subset H^1(\Omega)$. Assume that*

$$\sup_n \int_\Omega \Big(\operatorname{dist}(x, \partial\Omega)^\alpha |\nabla u_n(x)|^2 + |u_n|^2 \Big) \, dx < +\infty. \tag{3.20}$$

Then (u_n) is relatively compact in $L^2(\Omega)$.

Proof. Without loss of generality, one can assume that $\alpha > 1$. By Lemma 6, we have

$$\int_\Omega \operatorname{dist}(x, \partial\Omega)^{\alpha-2} |u_n(x)|^2 \, dx \le C_{\alpha,\Omega} \int_\Omega \Big(\operatorname{dist}(x, \partial\Omega)^\alpha |\nabla u_n(x)|^2 + |u_n(x)|^2 \Big) \, dx. \tag{3.21}$$

In this proof, $C_{\alpha,\Omega}$ denotes a positive constant depending only on α and Ω and can be changed from one place to another. We derive from (3.20) and (3.21) that, for $\tau > 0$ small,

$$\int_{\Omega_\tau} |u_n(x)|^2 \, dx \le \tau^{2-\alpha} \int_\Omega \Big(\operatorname{dist}(x, \partial\Omega)^\alpha |\nabla u_n(x)|^2 + |u_n|^2 \Big) \, dx \le C_{\alpha,\Omega} \tau^{2-\alpha}. \tag{3.22}$$

Fix $\varepsilon > 0$ arbitrary. Let $\tau > 0$ (small) be such that

$$\|u_n\|_{L^2(\Omega_\tau)} \le \varepsilon/2 \quad \forall n \in \mathbb{N}. \tag{3.23}$$

Such a τ exists by (3.22). From (3.20) and Rellich–Kondrachov's compactness criterion, see, e.g., [12], there exist $u_{n_1}, \cdots, u_{n_k}$ such that

$$\Big\{ u_n \in L^2(\Omega \setminus \Omega_\tau); \; n \in \mathbb{N} \Big\} \subset \bigcup_{j=1}^k \Big\{ u \in L^2(\Omega \setminus \Omega_\tau); \|u - u_{n_j}\|_{L^2(\Omega\setminus\Omega_\tau)} \le \varepsilon/2 \Big\}. \tag{3.24}$$

A combination of (3.23) and (3.24) yields

$$\{ u_n \in L^2(\Omega); \; n \in \mathbb{N} \} \subset \bigcup_{j=1}^k \{ u \in L^2(\Omega); \|u - u_{n_j}\|_{L^2(\Omega)} \le \varepsilon \}.$$

Therefore, (u_n) is relatively compact in $L^2(\Omega)$. □

We end this section with the following lemma which implies the uniqueness statement in Theorem 2.

Lemma 8. *Let F be a reflection from $U \setminus D$ to D_τ for some small $\tau > 0$ and for some smooth open subset U of $\mathbb{R}^d$ with $\bar{D} \subset U$. Assume that $u_0 \in H^1_{loc}(\mathbb{R}^d \setminus \Gamma) \cap L^2_{loc}(\mathbb{R}^d)$ is an outgoing solution to*

$$\operatorname{div}(s_0 A \nabla u_0) + k^2 s_0 \Sigma u_0 = 0 \text{ in } \mathbb{R}^d \setminus \Gamma, \tag{3.25}$$

such that the LHS of (3.3) is finite with $v_0 := u_0 \circ F^{-1}$ in D_τ,

$$u_0 - v_0 = 0 \quad \text{and} \quad (F_* A \nabla v_0 - A \nabla u_0 \big|_D) \cdot \nu = 0 \text{ on } \Gamma \tag{3.26}$$

and

$$\lim_{t\to 0_+} \Im\Big\{ \int\limits_{\partial D_t\setminus\Gamma} \big(F_* A\nabla v_0\cdot \nu \bar{v}_0 - A\nabla u_0 \cdot \nu \bar{u}_0\big)\Big\} = 0. \tag{3.27}$$

Then $u_0 = 0$ *in* $\mathbb{R}^d$.

Proof. Fix $R > R_0$. Multiplying (3.25) by $\bar{u}_0$ and integrating on $B_R \setminus \big(D \cup F^{-1}(D_t)\big)$ and $D \setminus D_t$ respectively, one has, for $0 < t < \tau$,

$$-\int\limits_{B_R\setminus(D\cup F^{-1}(D_t))} \langle A\nabla u_0, \nabla u_0\rangle + k^2 \int\limits_{B_R} \Sigma|u_0|^2 + \int\limits_{\partial B_R} \partial_r u_0 \bar{u}_0 + \int\limits_{[\partial F^{-1}(D_t)]\setminus\Gamma} A\nabla u_0\cdot\nu\, \bar{u}_0 = 0 \tag{3.28}$$

and

$$\int\limits_{D\setminus D_t} \langle A\nabla u_0, \nabla u_0\rangle - k^2 \int\limits_{D\setminus D_t} \Sigma|u_0|^2 - \int\limits_{\partial D_t\setminus\Gamma} A\nabla u_0\cdot\nu\, \bar{u}_0 = 0. \tag{3.29}$$

Here ν denotes the normal unit vector directed to the exterior of the set in which one integrates. Set

$$v_0 = u_0 \circ F^{-1} \text{ in } D_\tau.$$

Then, by [26, Lemma 2],

$$\int\limits_{\partial F^{-1}(D_t)\setminus\Gamma} A\nabla u_0\cdot\nu\, \bar{u}_0 = -\int\limits_{\partial D_t\setminus\Gamma} F_* A\nabla v_0\cdot\nu\, \bar{v}_0.$$

It follows from (3.27) that

$$\lim_{t\to 0} \Im\left[\int\limits_{\partial F^{-1}(D_t)\setminus\Gamma} A\nabla u_0\cdot\nu\, \bar{u}_0 + \int\limits_{\partial D_t\setminus\Gamma} A\nabla u_0\cdot\nu\, \bar{u}_0\right] = 0. \tag{3.30}$$

Subtracting (3.29) from (3.28), letting $t \to 0$, and using (3.30), we obtain

$$\Im\Big\{\int\limits_{\partial B_R} \partial_r u_0 \bar{u}_0\Big\} = 0.$$

This implies, by Rellich's lemma,

$$u_0 = 0 \text{ in } \mathbb{R}^d \setminus B_{R_0}.$$

Using (3.26) and the unique continuation principle, we reach

$$u_0 = 0 \text{ in } B_{R_0}.$$

Hence $u_0 = 0$ in $\mathbb{R}^d$. The proof is complete. $\square$

3.2. Proof of Theorem 2

The uniqueness of u_0 follows from Lemma 8. We next estimate u_δ. By Lemma 2,

$$\|u_\delta\|_{H^1(\mathbb{R}^d)} \le \frac{C}{\delta}\|f\|_{L^2(\mathbb{R}^d)}. \tag{3.31}$$

We prove by contradiction that

$$\|u_\delta\|_{L^2(B_{R_0})} \le C\|f\|_{L^2(\mathbb{R}^d)}. \tag{3.32}$$

Suppose that this is not true. There exist $\delta_n \to 0_+$, $f_n \in L^2(\mathbb{R}^d)$ with $\operatorname{supp} f_n \subset\subset B_{R_0}$ such that

$$\|u_{\delta_n}\|_{L^2(B_{R_0})} = 1 \quad \text{and} \quad \|f_n\|_{L^2(\mathbb{R}^d)} \to 0. \tag{3.33}$$

Here $u_{\delta_n} \in H^1(\mathbb{R}^d)$ is the unique solution of (1.3) with $\delta = \delta_n$ and $f = f_n$. Using (2.10) in Lemma 2, we have

$$\|u_\delta\|_{H^1(\mathbb{R}^d)} \le C\delta_n^{-1/2}. \tag{3.34}$$

We derive from Lemma 4 that

$$\operatorname{div}(F_*A\nabla v_{\delta_n}) + k^2 F_*\Sigma v_{\delta_n} + i\delta_n F_*1 v_{\delta_n} = F_*f_n \text{ in } D_\tau,$$

and

$$v_{\delta_n} = u_{\delta_n} \quad \text{and} \quad F_*A\nabla v_{\delta_n}\cdot\nu = (1+i\delta_n)A\nabla u_{\delta_n}\big|_D\cdot\nu \text{ on } \Gamma. \tag{3.35}$$

We also have

$$\operatorname{div}(A\nabla u_{\delta_n}) + k^2\Sigma u_\delta + (s_\delta^{-1}s_0 - 1)k^2\Sigma u_{\delta_n} + s_\delta^{-1} i\delta_n u_{\delta_n} = s_\delta^{-1} f \text{ in } D_\tau.$$

From (3.33), we derive that

$$\|\big(u_{\delta_n}, v_{\delta_n}\big)\|_{H^{1/2}(\partial D_{\tau/2}\setminus\Gamma)},\ \|\big(A\nabla u_{\delta_n}\cdot\nu, F*A\nabla v_{\delta_n}\cdot\nu\big)\|_{H^{-1/2}(\partial D_{\tau/2}\setminus\Gamma)} \text{ are bounded.} \tag{3.36}$$

Applying Lemma 5 with $D = D_{\tau/2}$ and using (3.33), (3.34), and (3.36), we obtain

$$\sup_n \int_{D_{\tau/2}} |\langle (A - F_*A)\nabla u_{\delta_n}, \nabla u_{\delta_n}\rangle| + |\nabla(u_{\delta_n} - v_{\delta_n})|^2 < +\infty.$$

By Lemma 7,

$$(u_{\delta_n}), (v_{\delta_n}) \text{ are relatively compact in } L^2(D_{\tau/2}).$$

This implies

$$\|\big(u_{\delta_n}, v_{\delta_n}\big)\|_{H^{1/2}(\partial D_{\tau/4}\setminus\Gamma)},\ \|\big(A\nabla u_{\delta_n}\cdot\nu, F*A\nabla v_{\delta_n}\cdot\nu\big)\|_{H^{-1/2}(\partial D_{\tau/4}\setminus\Gamma)} \text{ are bounded.}$$

From Lemmas 1 and 3, one may assume that

$$(u_{\delta_n}) \text{ converges in } L^2_{loc}(\mathbb{R}^d),$$

and (u_{δ_n}) and (v_{δ_n}) converge almost everywhere. Let u_0 be the limit of $\big(u_{\delta_n}\big)$ in $L^2_{loc}(\mathbb{R}^d)$ and v_0 be the limit of $\big(v_{\delta_n}\big)$ in $L^2(D_\tau)$. Then $u_0 \in H^1_{loc}(\mathbb{R}^d \setminus \Gamma) \cap L^2_{loc}(\mathbb{R}^d)$ is a solution to

$$\operatorname{div}(s_0 A \nabla u_0) + k^2 s_0 \Sigma u_0 = 0 \text{ in } \mathbb{R}^d \setminus \Gamma,$$

u_0 satisfies the outgoing condition by the limiting absorption principle, and $v_0 = u_0 \circ F^{-1}$ in D_τ. From (3.31) and (3.35), we obtain

$$u_0 - v_0 = 0 \quad \text{ and } \quad (A\nabla u_0\big|_D - F_* A \nabla v_0) \cdot \nu = 0 \text{ on } \Gamma,$$

and

$$\|u_0\|_{L^2(B_R)} + \|u_0 - v_0\|_{H^1(D_\tau)} + \Big(\int_{D_\tau} \big|\langle (A - F_*A)\nabla u_0, \nabla u_0\rangle\big|\Big)^{1/2} < +\infty.$$

Multiplying the equation of v_δ and u_δ by $\bar{v}_\delta$ and $\bar{u}_\delta$ respectively, integrating on D_τ, and considering the imaginary part, we have

$$\begin{aligned}
&\Im\Big\{ \int_{\partial D_t \setminus \Gamma} \big(F_* A \nabla v_\delta \cdot \nu \bar{v}_\delta - (1 + i\delta) A \nabla u_\delta \cdot \nu \bar{u}_\delta\big) + \int_{D_\tau} i\delta\Big(F_* 1 |v_\delta|^2 + \langle A \nabla u_\delta, \nabla u_\delta\rangle + |u_\delta|^2\Big)\Big\} \\
&= \Im\Big\{ \int_{D_t} F_* f \bar{v}_\delta + \int_{D_t} f \bar{u}_\delta \Big\}.
\end{aligned}$$

Letting $\delta \to 0$, we obtain

$$\Im\Big\{ \int_{\partial D_t \setminus \Gamma} \big(F_* A \nabla v_0 \cdot \nu \bar{v}_0 - A \nabla u_0 \cdot \nu \bar{u}_0\big)\Big\} = \Im\Big\{ \int_{D_t} F_* f \bar{v}_0 + \int_{D_t} f \bar{u}_0 \Big\}.$$

It follows that

$$\lim_{t \to 0} \Im\Big\{ \int_{\partial D_t \setminus \Gamma} \big(F_* A \nabla v_0 \cdot \nu \bar{v}_0 - A \nabla u_0 \cdot \nu \bar{u}_0\big)\Big\} = 0. \tag{3.37}$$

Hence $u_0 = 0$ by Lemma 8; this contradicts the fact $\|u_0\|_{L^2(B_{R_0})} = 1$ by (3.33). Estimate (3.32) is proved. Estimate (3.2) now follows from Lemma 5. Hence, as above, for any sequence $(\delta_n) \to 0$, there exists a subsequence (δ_{n_k}) such that $(u_{\delta_{n_k}})$ converges to u_0 weakly in $H^1_{loc}(\mathbb{R}^d \setminus \Gamma)$ and strongly in $L^2_{loc}(\mathbb{R}^d)$. It is clear that $u_0 \in H^1_{loc}(\mathbb{R}^d \setminus \Gamma) \cap L^2_{loc}(\mathbb{R}^d)$, $u_0 - v_0 \in H^1(D_\tau)$, and u_0 is the unique outgoing condition to

$$\operatorname{div}(s_0 A \nabla u_0) + k^2 s_0 \Sigma u_0 = f \text{ in } \mathbb{R}^d.$$

Since the limit u_0 is unique, the convergence holds as $\delta \to 0$. It is clear that estimate (3.3) is a direct consequence of (3.2). The proof is complete. □

3.3. Proof of Corollary 3

The proof of Corollary 3 is based on a reflection which is different from the standard one used in Corollary 2. Let F be defined as follows:

$$x_\Gamma - t\nu(x_\Gamma) \mapsto x_\Gamma + t\big[1 + tc(x_\Gamma)\big]\nu(x_\Gamma),$$

for $x_\Gamma \in \Gamma$ and $t > 0$ (small). Here $c(x_\Gamma) = \beta \operatorname{trace}\Pi(x_\Gamma)$ where $\Pi(x_\Gamma)$ is the second fundamental form of Γ at x_Γ and β is a constant defined later. In this proof, $\nu(x_\Gamma)$ denotes the unit normal vector of Γ at x_Γ directed into D. Fixing $x_\Gamma \in \Gamma$, we estimate $F_*A - A$ at $x_\Gamma + t\big[1 + tc(x_\Gamma)\big]\nu(x_\Gamma)$ for small positive t. To this end, we use local coordinates. Without loss of generality, one may assume that $x_\Gamma = 0$ and around $x_\Gamma = 0$, Γ is presented by the graph of a function $\varphi : (-\varepsilon_0, \varepsilon_0)^{d-1} \to \mathbb{R}$ with $\varphi(0) = 0$, and $\{(x', x_d) \in (-\varepsilon_0, \varepsilon_0)^d; x_d > \varphi(x')\} \subset D$. We also assume that $\nabla'\varphi'(0) := (\partial_{x_1}\varphi, \cdots, \partial_{x_{d-1}}\varphi)(0) = 0 \in \mathbb{R}^{d-1}$ and $\nabla'^2\varphi(0) = \lambda_1 e_1 \otimes e_1 + \cdots \lambda_{d-1} e_{d-1} \otimes e_{d-1}$ where $\lambda_1, \cdots, \lambda_{d-1}$ are the eigenvalues of $\Pi(x_\Gamma)$. Here $e_1, \cdots, e_d$ is an orthogonal basis of $\mathbb{R}^d$. Since Γ is strictly convex, one can assume that φ is strictly convex or strictly concave. We only consider the case φ is strictly convex; the other case can be proceeded similarly. Hence, in what follows, we assume that $\lambda_i > 0$ for $1 \le i \le d-1$. Set

$$\boldsymbol{\varphi}(x', t) = \varphi(x', 0).$$

Define

$$G_1(x', t) = (x', \varphi(x')) + \frac{t\big[1 + tc(x')\big]}{\sqrt{1 + |\nabla_{x'}\varphi(x')|^2}}\Big(- \nabla_{x'}\varphi(x'), 1\Big).$$

A computation yields

$$\nabla G_1(0, t) = I - t\nabla^2\boldsymbol{\varphi}(0) + 2tc(x')e_d \otimes e_d + O(t^2). \tag{3.38}$$

Here and in what follows in this paper, $O(s)$ denotes a quantity or a matrix whose norm is bounded by $C|s|$ for some positive constant C independent of s for small s. Define

$$G_2(x', t) = (x', \varphi(x')) - \frac{t}{\sqrt{|\nabla'\varphi(x')|^2 + 1}}\Big(- \nabla'\varphi(x'), 1\Big).$$

We have

$$\nabla G_2(0, t) = I - 2e_d \otimes e_d + t\nabla^2\boldsymbol{\varphi}(0). \tag{3.39}$$

From the definition of F, G_1, and G_2, we have

$$F(y) = G_1 \circ G_2^{-1}(y).$$

This yields

$$\nabla F(y) = \nabla G_1(x', t)[\nabla G_2(x', t)]^{-1} \text{ where } G_2(x', t) = y.$$

We derive from (3.38) and (3.39) that

$$\nabla F(y) = I - 2e_d \otimes e_d - 2t\Pi - 2tc(0)\, e_d \otimes e_d + O(t^2),$$

for $y = G_2(0, t)$. Here for notational ease, we also denote $\Pi = \nabla^2\boldsymbol{\varphi}(0)$. We have, for $y = G_2(0, t)$,

$$\begin{aligned} |\det \nabla F(y)|^{-1}\nabla F(y)^T\nabla F(y) &= \big[1 + 2t \operatorname{trace}\Pi - 2tc(0)\big]\big(I - 4t\Pi + 4tc(0)e_d \otimes e_d\big) + O(t^2) \\ &= I + 2t\sum_{i=1}^{d-1}\big[\operatorname{trace}\Pi - 2\lambda_i - c(0)\big]e_i \otimes e_i + 2t\big[c(0) + \operatorname{trace}\Pi\big]e_d \otimes e_d + O(t^2). \end{aligned}$$

By taking $c(0) = \beta \mathrm{trace}\Pi$ with $-1 < \beta < 0$ and β is closed to -1, we have

$$B := |\det \nabla F(y)|^{-1} \nabla F(y)^T \nabla F(y) - I \geq \gamma t I,$$

for some $\gamma > 0$. The conclusion now follows from Theorem 2. The proof is complete. □

Remark 11. Corollary 3 does not hold for $d = 2$. Indeed, assume that $A = I$ in $\mathbb{R}^2$, $D = B_{r_2} \setminus B_{r_1}$ for $0 < r_1 < r_2$. Let $F : B_{r_2^2/r_1} \setminus B_{r_2} \to B_{r_2} \setminus B_{r_1}$ be the Kelvin transform with respect to ∂B_{r_2} and let $\Sigma = F_*1$ in $B_{r_2} \setminus B_{r_1}$, then $F_*A = A$ and $F_*\Sigma = \Sigma$: the resonance appears (Proposition 2 in Section 5). The strict convexity condition of Γ is necessary in three dimensions. In fact assume that $D = \{(x_1, x_2, x_3) \in \mathbb{R}^3;\ x_1^2 + x_2^2 < 1$ and $0 < x_3 < 1\}$ and let $G : \mathbb{R}^2 \times (0,1) \setminus D \to D$ be defined by $G(x_1, x_2, x_3) = \big(F(x_1, x_2), x_3\big)$. Set $(A, \Sigma) = (I, 1)$ in $\mathbb{R}^3 \setminus D$ and (I, G_*1) otherwise. The problem is not well-posed again for some f by Proposition 2 in Section 5. Nevertheless, the strict convexity condition can be weaken in four or higher dimensions. To illustrate this point, let consider the case $d = 4$. Then

$$\begin{aligned}\frac{1}{2t} B = {} & (\lambda_2 + \lambda_3 - \lambda_1 - \beta) e_1 \otimes e_1 + (\lambda_1 + \lambda_3 - \lambda_2 - \beta) e_e \otimes e_2 \\ & + (\lambda_1 + \lambda_2 - \lambda_2 - \beta) e_3 \otimes e_3 + (1 + \beta)(\lambda_1 + \lambda_2 + \lambda_3) e_4 \otimes e_4 + O(t).\end{aligned}$$

Assume that $\lambda_1, \lambda_2, \lambda_3 \geq 0$ and if $\lambda_1\lambda_2\lambda_3 = 0$ then only one of them is 0. Then $B \geq \gamma t I$ if β is chosen as in the proof of Corollary 3. Hence the conclusion of Corollary 3 holds in this case.

4. A variational approach via the multiplier technique

In this section we develop a variational approach via the multiplier technique to deal with the case $F_*A = A$ in D_τ. This complements the results in the previous sections. The main result of this section is:

Theorem 3. *Let $f \in L^2(\mathbb{R}^d)$ with $\operatorname{supp} f \subset B_{R_0} \setminus \Gamma$, and let $u_\delta \in H^1(\mathbb{R}^d)$ $(0 < \delta < 1)$ be the unique solution of (1.3). Assume that there exists a reflection F from $U \setminus D$ to D_τ, for some $\tau > 0$ and for some smooth bounded open subset U of $\mathbb{R}^d$ with $\bar{D} \subset U$ such that either*

$$F_*A - A \geq 0 \quad \text{and} \quad \Sigma - F_*\Sigma \geq c \operatorname{dist}(x, \Gamma)^\beta, \tag{4.1}$$

or

$$A - F_*A \geq 0 \quad \text{and} \quad F_*\Sigma - \Sigma \geq c \operatorname{dist}(x, \Gamma)^\beta, \tag{4.2}$$

in each connected component of D_τ, for some $\beta > 0$ and $c > 0$. Set $v_\delta = u_\delta \circ F^{-1}$ in D_τ. Then, for all $0 < \rho < R$,

$$\begin{aligned}\int_{B_R \setminus (D_\rho \cup D_{-\rho})} |u_\delta|^2 + \int_{D_\tau} |\Sigma - F_*\Sigma||u_\delta|^2 + \int_{D_\tau} |\langle (A - F_*A)\nabla u_\delta, \nabla u_\delta\rangle| \\ + \int_{D_\tau} |u_\delta - v_\delta|^2 + |\nabla(u_\delta - v_\delta)|^2 \leq C_{R,\rho} \|f\|^2_{L^2(\mathbb{R}^d)}.\end{aligned} \tag{4.3}$$

Moreover, (u_δ) converges to u_0 weakly in $H^1_{loc}(\mathbb{R}^d \setminus \Gamma)$ and strongly in $L^2_{loc}(\mathbb{R}^d \setminus \Gamma)$ as $\delta \to 0$, where $u_0 \in H^1_{loc}(\mathbb{R}^d \setminus \Gamma)$ is the **unique** *outgoing solution of (1.2) such that the LHS of (4.4) is finite, where $v_0 := u_0 \circ F^{-1}$ in D_τ. Consequently,*

$$\int_{B_R\setminus(D_\rho\cup D_{-\rho})} |u_0|^2 + \int_{D_\tau} |\Sigma - F_*\Sigma||u_0|^2 + \int_{D_\tau} |\langle (A - F_*A)\nabla u_0, \nabla u_0\rangle|$$
$$+ \int_{D_\tau} |u_0 - v_0|^2 + |\nabla(u_0 - v_0)|^2 \le C_{R,\rho}\|f\|^2_{L^2(\mathbb{R}^d)}. \tag{4.4}$$

Here $C_{R,\rho}$ denotes a positive constant depending on R, ρ, A, Σ, R_0, β, c, and the distance between $\operatorname{supp} f$ *and Γ, but independent of f and δ.*

The solution u_0 in Theorem 3 is not in $L^2_{loc}(\mathbb{R}^d)$. Its meaning is given in the following definition:

Definition 5. Let $f \in L^2(\mathbb{R}^d)$ with $\operatorname{supp} f \subset\subset \mathbb{R}^d \setminus \Gamma$, and let F be a reflection from $U \setminus D$ to D_τ for some $\tau > 0$ and for some smooth open subset U of $\mathbb{R}^d$ with $\bar{D} \subset U$ such that (4.1) or (4.2) holds. A function $u_0 \in H^1_{loc}(\mathbb{R}^d \setminus \Gamma)$ such that the LHS of (4.4) is finite is called a solution to (1.2) if, with $v_0 = u_0 \circ F^{-1}$,

$$\operatorname{div}(s_0 A\nabla u_0) + k^2 s_0 \Sigma u_0 = f \text{ in } \mathbb{R}^d \setminus \Gamma, \tag{4.5}$$

$$u_0 - v_0 = 0 \quad \text{and} \quad (F_*A\nabla v_0 - A\nabla u_0\big|_D)\cdot \nu = 0 \text{ on } \Gamma, \tag{4.6}$$

and

$$\lim_{t\to 0_+} \Im\Big\{ \int_{\partial D_t\setminus\Gamma} \big(F_*A\nabla v_0\cdot \nu\bar{v}_0 - A\nabla u_0\cdot\nu\bar{u}_0\big)\Big\} = 0. \tag{4.7}$$

Remark 12. Since $u_0 - v_0 \in H^1(D_\tau)$, $(\Sigma - F_*\Sigma)u_0 \in L^2(D_\tau)$, and $(A - F_*A)\nabla u_0 \in L^2(D_\tau)$, it follows that $\operatorname{div}(F_*A\nabla v_0 - A\nabla u_0) \in L^2(D_\tau)$ and $F_*A\nabla v_0 - A\nabla u_0 \in L^2(D_\tau)$. Hence $u_0 - v_0 \in H^{1/2}(\Gamma)$, and $(F_*A\nabla v_0 - A\nabla u_0\big|_D)\cdot\nu \in H^{-1/2}(\Gamma)$. Requirement (3.5) makes sense.

Remark 13. β is only required to be positive in Theorem 3. In (4.1) and (4.2) of Theorem 3, we only make the assumption on the lower bound and not on the upper bound of the quantities considered.

The proof of Theorem 3 is based on a variational approach via the multiplier technique. One of the key point of the proof is Lemma 9, a variant of Lemma 5, where test functions are used. Sylvester in [43] used related ideas to study the transmission eigenvalues problem. The compactness argument used in the proof of Theorem 3 is different from the standard one used in the proof of Theorem 2 due to fact the family (u_δ) might not be bounded in $L^2_{loc}(\mathbb{R}^d)$ in the context considered in Theorem 3.

Here is a corollary of Theorem 3 which is a complement to Corollary 3 in two dimensions.

Corollary 4. *Let $d = 2$, $\sigma_0 \in \mathbb{R}$, $D = B_1$, $f \in L^2(\mathbb{R}^d)$ with* $\operatorname{supp} f \subset B_{R_0} \setminus \Gamma$*, and let $u_\delta \in H^1(\mathbb{R}^d)$ $(0 < \delta < 1)$ be the unique solution of (1.3). Assume that $(A, \Sigma) = (I, 1)$ in $D_{-\tau}$ and $(A, \Sigma) = (I, \sigma_0)$ in D_τ for some small $\tau > 0$. Let F be the Kelvin transform with respect to ∂D. Then $F_*I = I$ in $D_{\tau/2}$ and $F_*1 - \sigma_0 \ge c\operatorname{dist}(x, \Gamma)$ in $D_{\tau/2}$ if $\sigma_0 \le 1$ and $\sigma_0 - F_*1 > c$ in $D_{\tau/2}$ if $\sigma_0 > 1$ for some $c > 0$. As a consequence, (u_δ) converges u_0 weakly in $H^1_{loc}(\mathbb{R}^2 \setminus \Gamma)$ and strongly in $L^2_{loc}(\mathbb{R}^2 \setminus \Gamma)$ as $\delta \to 0$, where $u_0 \in H^1_{loc}(\mathbb{R}^d \setminus \Gamma)$ is the unique outgoing solution of (1.2); moreover, u_0 satisfies (4.4).*

The rest of this section contains three subsections and is devoted to the proof of Theorem 3 and Corollary 4. The first one is on a variant of Lemma 5 used in the proof of Theorem 3. The proof of Theorem 3 and Corollary 4 are given in the last two subsections.

4.1. A useful lemma

The following lemma is a variant of Lemma 5 and plays an important role in the proof of Theorem 3.

Lemma 9. *Let Ω be a smooth bounded open subset of $\mathbb{R}^d$, and A_1 and A_2 be two symmetric uniformly elliptic matrices, and Σ_1 and Σ_2 be two bounded real functions defined in Ω. Let $f_1, f_2 \in L^2(\Omega)$, $h \in H^{-1/2}(\partial\Omega)$, and let $u_1, u_2 \in H^1(\Omega)$ be such that*

$$\operatorname{div}(A_1 \nabla u_1) + \Sigma_1 u_1 = f_1 \quad \text{and} \quad \operatorname{div}(A_2 \nabla u_2) + \Sigma_2 u_2 = f_2 \text{ in } \Omega, \tag{4.8}$$

and

$$u_1 = u_2 \quad \text{and} \quad A_1 \nabla u_1 \cdot \nu = A_2 \nabla u_2 \cdot \nu + h \text{ on } \partial\Omega. \tag{4.9}$$

Assume that

$$A_1 \geq A_2 \quad \text{and} \quad \Sigma_2 \geq \Sigma_1 \text{ in } \Omega. \tag{4.10}$$

We have

$$\int_\Omega (\Sigma_2 - \Sigma_1)|u_2|^2 + \langle (A_1 - A_2)\nabla u_2, \nabla u_2 \rangle + |\nabla(u_1 - u_2)|^2 \leq C\mathcal{N}(f_1, f_2, h, u_1, u_2),$$

for some positive constant C independent of u_1, u_2, f_1, f_2, and h, where

$$\mathcal{N}(f_1, f_2, h, u_1, u_2) = \|(u_1, u_2)\|_{L^2(\Omega)} \|(f_1, f_2)\|_{L^2(\Omega)} + \|h\|_{H^{-1/2}(\partial\Omega)} \|(u_1, u_2)\|_{H^{1/2}(\partial\Omega)} + \|u_1 - u_2\|^2_{L^2(\Omega)}.$$

Proof. By considering the real part and the imaginary part separately, without loss of generality, one may assume that all functions mentioned in Lemma 9 are real. Define

$$w = u_1 - u_2 \text{ in } \Omega.$$

From (4.8) and (4.9), we have

$$\operatorname{div}(A_1 \nabla w) + \Sigma_1 w = f_1 - f_2 + (\Sigma_2 - \Sigma_1) u_2 + \operatorname{div}([A_2 - A_1]\nabla u_2) \text{ in } \Omega, \tag{4.11}$$

$$w = 0, \quad \text{and} \quad A\nabla w \cdot \nu = h \text{ on } \partial\Omega. \tag{4.12}$$

Multiplying (4.11) by u_2 and integrating on Ω, we have

$$\int_\Omega (f_1 - f_2) u_2 + (\Sigma_2 - \Sigma_1)|u_2|^2 + \int_\Omega \langle (A_1 - A_2)\nabla u_2, \nabla u_2 \rangle = \int_\Omega \Big(\operatorname{div}(A_1 \nabla w) + \Sigma_1 w \Big) u_2 + \int_{\partial\Omega} (A_1 - A_2)\nabla u_2 \cdot \nu \, u_2.$$

Integrating by parts and using the fact that

$$A_1 \nabla w \cdot \nu + (A_1 - A_2)\nabla u_2 \cdot \nu = A_1 \nabla u_1 \cdot \nu - A_2 \nabla u_2 \cdot \nu = h \text{ on } \partial\Omega,$$

$$\int_\Omega A_1 \nabla w \nabla u_2 = \int_\Omega A_2 \nabla w \nabla u_2 + \int_\Omega (A_1 - A_2)\nabla w \nabla u_2,$$

and

$$\Sigma_1 w u_2 = (\Sigma_1 - \Sigma_2) w u_2 + \Sigma_2 w u_2,$$

we derive from (4.10) and (4.12) that

$$\int_\Omega (\Sigma_2 - \Sigma_1)|u_2|^2 + \int_\Omega \langle (A_1 - A_2)\nabla u_2, \nabla u_2\rangle \\ \le C\mathcal{N}(f_1, f_2, h, u_1, u_2) + \int_\Omega (\Sigma_1 - \Sigma_2) w u_2 + \int_\Omega \langle (A_2 - A_1)\nabla w, \nabla u_2\rangle. \tag{4.13}$$

Here and in what follows in this proof, C denotes a positive constant independent of f_j, h, u_j for $j = 1, 2$. Multiplying (4.11) by w and integrating on Ω, we have

$$\int_\Omega \langle A_1 \nabla w, \nabla w\rangle \le C\mathcal{N}(f_1, f_2, h, u_1, u_2) + \int_\Omega (\Sigma_1 - \Sigma_2) w u_2 + \int_\Omega \langle (A_2 - A_1)\nabla u_2, \nabla w\rangle. \tag{4.14}$$

A combination of (4.13) and (4.14) yields

$$\int_\Omega (\Sigma_2 - \Sigma_1)|u_2|^2 + \int_\Omega \langle (A_1 - A_2)\nabla u_2, \nabla u_2\rangle + \int_\Omega \langle A_1 \nabla w, \nabla w\rangle \\ \le C\mathcal{N}(f_1, f_2, h, u_1, u_2) + 2\int_\Omega (\Sigma_1 - \Sigma_2) w u_2 + 2\int_\Omega \langle (A_2 - A_1)\nabla w, \nabla u_2\rangle. \tag{4.15}$$

We have for $\lambda > 0$, since $A_1 \ge A_2$,

$$2\int_\Omega \langle (A_2 - A_1)\nabla u_2, \nabla w\rangle \le \lambda \int_\Omega \langle (A_1 - A_2)\nabla u_2, \nabla u_2\rangle + \lambda^{-1}\langle (A_1 - A_2)\nabla w, \nabla w\rangle, \tag{4.16}$$

and, since $\Sigma_2 \ge \Sigma_1$,

$$2\int_\Omega (\Sigma_1 - \Sigma_2) w u_2 \le 2\int_\Omega (\Sigma_2 - \Sigma_1) w^2 + \frac{1}{2}\int_\Omega (\Sigma_2 - \Sigma_1) u_2^2. \tag{4.17}$$

By choosing λ smaller than 1 and close to 1, we derive from (4.15), (4.16), and (4.17) that

$$\int_\Omega (\Sigma_2 - \Sigma_1)|u_2|^2 + \int_\Omega \langle (A_1 - A_2)\nabla u_2, \nabla u_2\rangle + \int_\Omega |\nabla w|^2 \le C\mathcal{N}(f_1, f_2, h, u_1, u_2). \tag{4.18}$$

The proof is complete. $\square$

4.2. Proof of Theorem 3

The proof of the uniqueness of u_0, i.e., if $f = 0$ then $u_0 = 0$ is similar to the one of Lemma 8. The details are left to the reader.

We next establish the estimate for u_δ by a compactness argument. The compactness argument used in this proof is different from the one in the proof of Theorem 2 due to the loss of the control of u_δ in $L^2_{loc}(\mathbb{R}^d)$. Without loss of generality, one may assume that $\operatorname{supp} f \cap (D_\tau \cup F^{-1}(D_\tau)) = \emptyset$. By Lemma 2, we have

$$\|u_\delta\|^2_{H^1(\mathbb{R}^d)} \le \frac{C}{\delta}\|f\|_{L^2(\mathbb{R}^d)}\|u_\delta\|_{L^2(B_{R_0}\setminus(D_\tau\cup F^{-1}(D_\tau)))}. \tag{4.19}$$

We first prove that

$$\|u_\delta\|_{L^2(B_{R_0}\setminus(D_{\tau_1}\cup F^{-1}(D_{\tau_1})))} \le C\|f\|_{L^2(\mathbb{R}^d)}, \tag{4.20}$$

by contradiction[5] where $0 < \tau_1 < \tau/3$ is a positive constant chosen later. Assume that there exist $\delta_n \to 0$, $f_n \in L^2(\mathbb{R}^d)$ with $\operatorname{supp} f_n \subset B_{R_0}$ and $\operatorname{supp} f_n \cap (D_\tau \cup F^{-1}(D_\tau)) = \emptyset$ such that

$$\|f_n\|_{L^2(\mathbb{R}^d)} \to 0 \quad \text{ and } \quad \|u_n\|_{L^2(B_{R_0}\setminus(D_{\tau_1}\cup F^{-1}(D_{\tau_1})))} = 1, \tag{4.21}$$

where u_n is the solution of (1.3) with $\delta = \delta_n$ and $f = f_n$. Set $v_n = u_n \circ F^{-1}$ in D_τ. By Lemma 4,

$$\operatorname{div}(F_*A\nabla v_n) + k^2 F_*\Sigma v_n + i\delta_n F_*1 v_n = 0 \text{ in } D_\tau, \tag{4.22}$$

and

$$v_n = u_n \quad \text{ and } \quad A\nabla v_n\big|_D \cdot \nu = F_*A\nabla u_n \cdot \nu + i\delta_n A\nabla u_n\big|_D \cdot \nu \text{ on } \Gamma. \tag{4.23}$$

We also have

$$\operatorname{div}(A\nabla u_n) + k^2\Sigma u_n + \Big(i\delta_n s_{\delta_n}^{-1} + [s_0 s_{\delta_n}^{-1} - 1]k^2\Sigma\Big)u_n = 0 \text{ in } D_\tau. \tag{4.24}$$

Applying Lemma 9 with $D = D_{\tau/2}$ and using (4.19) and (4.21), we have

$$\int\limits_{D_{\tau/2}} |\Sigma - F_*\Sigma||u_n|^2 + \int\limits_{D_{\tau/2}} |\langle (A - F_*A)\nabla u_n, \nabla u_n\rangle| + \int\limits_{D_{\tau/2}} |\nabla(u_n - v_n)|^2 \le C_\tau\Big(1 + \int\limits_{D_{\tau_1}} |u_n - v_n|^2\Big). \tag{4.25}$$

By choosing τ_1 small enough, one has

$$C_\tau \int\limits_{D_{\tau_1}} |u_n - v_n|^2 \le \frac{1}{2}\int\limits_{D_{\tau/2}} |\nabla(u_n - v_n)|^2,$$

since $u_n - v_n = 0$ on Γ. It follows from (4.25) that

$$\int\limits_{D_{\tau/2}} |\Sigma - F_*\Sigma||u_n|^2 + \int\limits_{D_{\tau/2}} |\langle (A - F_*A)\nabla u_n, \nabla u_n\rangle| + \int\limits_{D_{\tau/2}} |\nabla(u_n - v_n)|^2 + \int\limits_{D_{\tau/2}} |u_n - v_n|^2 \le C_\tau. \tag{4.26}$$

This implies, by (4.1) and (4.2), for $0 < \rho < \tau/4$,

$$\|(u_n, v_n)\|_{H^{1/2}(\partial D_\rho\setminus\Gamma)},\ \|(A\nabla u_n\cdot\nu, F * A\nabla v_n\cdot\nu)\|_{H^{-1/2}(\partial D_\rho\setminus\Gamma)} \text{ are bounded.}$$

[5] We do not prove that $\|u_\delta\|_{L^2(B_{R_0})} \le C\|f\|_{L^2(\mathbb{R}^d)}$. This is different from the proof of Theorem 2.

Using Lemmas 1 and 3, we derive that

$$\int\limits_{B_R\setminus(D_\rho\cup D_{-\rho})} |u_n|^2 + |\nabla u_n|^2 \le C_{\rho,R}, \tag{4.27}$$

for $0 < \rho < R$. Without loss of generality, one may assume that (u_n) converges to u_0 weakly in $H^1_{loc}(\mathbb{R}^d\setminus\Gamma)$, and strongly in $L^2_{loc}(\mathbb{R}^d\setminus\Gamma)$, (v_n) converges to v_0 weakly in $H^1_{loc}(D_\tau)$ and strongly in $L^2_{loc}(D_\tau)$, and $(u_n - v_n)$ converges to $u_0 - v_0$ weakly in $H^1(D_\tau)$ and strongly in $L^2(D_\tau)$, and $v_0 = u_0 \circ F^{-1}$ in D_τ. We have, by (4.26),

$$\int\limits_{D_\tau} |\Sigma - F_*\Sigma||u_0|^2 + \int\limits_{D_\tau} |\langle (A - F_*A)\nabla u_0, \nabla u_0\rangle|$$
$$+ \int\limits_{D_\tau} |u_0 - v_0|^2 + |\nabla(u_0 - v_0)|^2 + \int\limits_{B_R\setminus(D_\rho\cup D_\rho)} |u_0|^2 + |\nabla u_0|^2 \le C_{\rho,R},$$

and $u_0 \in H^1_{loc}(\mathbb{R}^d\setminus\Gamma)$ is an outgoing solution to the equation

$$\operatorname{div}(s_0 A\nabla u_0) + k^2 s_0 \Sigma u_0 = 0 \text{ in } \mathbb{R}^d\setminus\Gamma.$$

From (4.19) and (4.23), we obtain

$$u_0 - v_0 = 0 \quad \text{ and } (A\nabla u_0\big|_D - F_*A\nabla v_0)\cdot\nu = 0 \text{ on } \Gamma.$$

Similar to (3.37), we also have

$$\lim_{t\to 0_+} \Im\Big\{ \int\limits_{\partial D_t\setminus\Gamma} \big(F_*A\nabla v_0\cdot\nu\bar{v}_0 - A\nabla u_0\cdot\nu\bar{u}_0\big)\Big\} = 0.$$

Hence $u_0 = 0$ in $\mathbb{R}^d$ by the uniqueness. We have a contradiction with the fact that

$$\|u_0\|_{L^2(B_{R_0}\setminus(D_{\tau_1}\cup F^{-1}(D_{\tau_1})))} = \lim_{n\to\infty} \|u_n\|_{L^2(B_{R_0}\setminus(D_{\tau_1}\cup F^{-1}(D_{\tau_1})))} = 1.$$

Claim (4.20) is proved. The conclusion now is standard as in the proof of Theorem 2. The details are left to the reader. □

4.3. Proof of Corollary 4

It suffices to check $F_*1 - \sigma_0 \ge c\,\mathrm{dist}(x,\Gamma)$ if $\sigma_0 \le 1$ and $\sigma_0 - F_*1 > c$ if $\sigma_0 > 1$ in $D_{\tau/2}$ for some $c > 0$ provided that τ is small enough. A computation gives

$$|\det(\nabla F)(y)| = 1 - 4\,\mathrm{dist}(x,\Gamma) + O\big(\mathrm{dist}(x,\Gamma)^2\big),$$

where $F(y) = x$. This implies

$$1/|\det(\nabla F)(y)| = 1 + 4\,\mathrm{dist}(x,\Gamma) + O\big(\mathrm{dist}(x,\Gamma)^2\big),$$

where $F(y) = x$. The conclusion follows from the definition of F_*1 and the fact $F_*I = I$. □

5. Optimality of the main results

In this section, we show that the system is resonant if the requirements on A and Σ mentioned in Theorems 1, 2, 3 are not fulfilled and Theorems 2 and 3 are "optimal". More precisely, we have

Proposition 2. *Assume that there exists a reflection $F : U \setminus \bar{D} \to D_\tau$ for some smooth open subset U of $\mathbb{R}^d$ with $\bar{D} \subset U$ and some $\tau > 0$ such that*

$$(A, \Sigma) = (F_*A, F_*\Sigma) \text{ in } B(x_0, \hat{r}_0) \cap D,$$

for some $x_0 \in \Gamma$ and $\hat{r}_0 > 0$. Let $f \in L^2(\mathbb{R}^d)$ with $\operatorname{supp} f \subset\subset B_{R_0} \setminus \Gamma$ and assume that A is Lipschitz in $\overline{D \cap B(x_0, \hat{r}_0)}$. There exists $0 < r_0 < \hat{r}_0$, independent of f, such that if there is no solution in $H^1(D \cap B(x_0, r_0))$ to the Cauchy problem:

$$\operatorname{div}(A\nabla w) + k^2\Sigma w = f \text{ in } D \cap B(x_0, r_0) \quad \text{and} \quad w = A\nabla w \cdot \nu = 0 \text{ on } \partial D \cap B(x_0, r_0),$$

then $\limsup_{\delta \to 0} \|u_\delta\|_{L^2(K)} = +\infty$ for some $K \subset\subset B_{R_0} \setminus \Gamma$ where $u_\delta \in H^1(\mathbb{R}^d)$ is the unique solution of (1.3).

Recall that $B(x, r)$ denotes the open ball centered at x and of radius r.

Proof. Without loss of generality, one may assume that $x_0 = 0$ and $\hat{r}_0$ is small. We prove Proposition 2 by contradiction. Assume that the conclusion is not true. Then even for small r_0, there exists f with $\operatorname{supp} f \cap B_{\hat{r}_0} \setminus \Gamma$ such that there is no solution in $H^1(D \cap B(x_0, r_0))$ to the Cauchy problem:

$$\operatorname{div}(A\nabla w) + k^2\Sigma w = f \text{ in } D \cap B(x_0, r_0) \quad \text{and} \quad w = A\nabla w \cdot \nu = 0 \text{ on } \partial D \cap B(x_0, r_0),$$

and

$$\limsup_{\delta \to 0} \|u_\delta\|_{L^2(K)} < +\infty \text{ for all } K \subset\subset B_{R_0} \setminus \Gamma.$$

Using Lemma 2, we have

$$\|u_\delta\|_{H^1(B_{R_0})} \le C\delta^{-1/2}, \tag{5.1}$$

since $\operatorname{supp} f \subset\subset B_{R_0} \setminus \Gamma$. Set $v_\delta = u_\delta \circ F^{-1}$ in $D \cap B(x_0, \hat{r}_0)$ and define $w_\delta = v_\delta - u_\delta$ in $D \cap B(x_0, \hat{r}_0)$. By Lemma 4, we have

$$\operatorname{div}(A\nabla v_\delta) + k^2\Sigma v_\delta = -i\delta_n F_*1 v_\delta \text{ in } D \cap B(x_0, \hat{r}_0).$$

Since

$$\operatorname{div}(A\nabla u_\delta) + k^2\Sigma u_\delta = k^2(1 - s_\delta^{-1}s_0)\Sigma u_\delta - i\delta s_\delta^{-1}u_\delta + s_\delta^{-1}f \text{ in } D \cap B(x_0, \hat{r}_0),$$

it follows that

$$\operatorname{div}(A\nabla w_\delta) + k^2\Sigma w_\delta = g_\delta \text{ in } D \cap B(x_0, \hat{r}_0),$$

where

$$g_\delta = f - i\delta F_*1 v_\delta - k^2(1 - s_\delta^{-1}s_0)\Sigma u_\delta + i\delta s_\delta^{-1}u_\delta - (s_\delta^{-1} + 1)f \text{ in } D \cap B(x_0, \hat{r}_0).$$

By Lemma 4, we also have

$$w_\delta = 0 \quad \text{and} \quad A\nabla w_\delta \cdot \nu = i\delta \nabla u_\delta\big|_D \cdot \nu \text{ on } \partial D \cap B(x_0, \hat{r}_0).$$

Using a local chart and applying Lemma 10 below, we have

$$\limsup_{\delta \to 0} \delta^{1/2} \| w_\delta \|_{H^1(D \cap B(x_0, \hat{r}_0))} = +\infty.$$

This contradicts (5.1). The proof is complete. □

The following lemma is used in the proof of Proposition 2.

Lemma 10. *Let $R > 0$, a be a Lipschitz symmetric uniformly elliptic matrix and σ be a real bounded function defined in $B_R \cap \mathbb{R}^d_+$, and let $g \in L^2(B_R)$. Assume that $W_\delta \in H^1(B_R \cap \mathbb{R}^d_+)$ $(0 < \delta < 1)$ satisfies*

$$\operatorname{div}(a\nabla W_\delta) + \sigma W_\delta = g_\delta \text{ in } B_R \cap \mathbb{R}^d_+,$$

$$W_\delta = 0 \text{ on } B_R \cap \mathbb{R}^d_0, \quad \text{and} \quad a\nabla W_\delta \cdot \nu = h_\delta \text{ on } B_R \cap \mathbb{R}^d_0,$$

for some $h_\delta \in H^{-1/2}(B_R \cap \mathbb{R}^d_0)$ such that

$$\| g_\delta - g \|_{L^2(B_R \cap \mathbb{R}^d_+)} + \| h_\delta \|_{H^{-1/2}(B_R \cap \mathbb{R}^d_0)} \le c\delta^{1/2}, \tag{5.2}$$

for some $c > 0$. There exists a constant $0 < r < R$ depending only on R, and the ellipticity and the Lipschitz constants of a, but independent of δ, c, g_δ, g, h_δ, and σ, such that if there is **no** *$W \in H^1(B_r \cap \mathbb{R}^d_+)$ with the properties*

$$\operatorname{div}(a\nabla W) + \sigma W = g \text{ in } B_R \cap \mathbb{R}^d_+, \quad W = 0 \text{ on } B_R \cap \mathbb{R}^d_0, \quad \text{and} \quad a\nabla W \cdot \nu = 0 \text{ on } B_R \cap \mathbb{R}^d_0, \tag{5.3}$$

then

$$\limsup_{\delta \to 0} \delta^{1/2} \| W_\delta \|_{H^1(B_R \cap \mathbb{R}^d_+)} = +\infty. \tag{5.4}$$

Here and in what follows, we denote $\mathbb{R}^d_+ = \mathbb{R}^d_{e_d,+}$ and $\mathbb{R}^d_0 = \mathbb{R}^d_{e_d,0}$ with $e_d = (0, \cdots, 0, 1) \in \mathbb{R}^d$.

Proof. For notational ease, $W_{2^{-n}}$, $g_{2^{-n}}$, and $h_{2^{-n}}$ are denoted by W_n, g_n, and h_n respectively. We have

$$\operatorname{div}(a\nabla W_n) + \sigma W_n = g_n \text{ in } B_R \cap \mathbb{R}^d_+,$$

$$W_n = 0 \text{ on } B_R \cap \mathbb{R}^d_0, \quad a\nabla W_n \cdot \nu = h_n \text{ on } B_R \cap \mathbb{R}^d_0.$$

We prove by contradiction that

$$\limsup_{n \to +\infty} 2^{-n/2} \| W_n \|_{H^1(B_R \cap \mathbb{R}^d_+)} = +\infty. \tag{5.5}$$

Assume that

$$m := \sup_n 2^{-n/2} \|W_n\|_{H^1(B_R \cap \mathbb{R}^d_+)} < +\infty. \tag{5.6}$$

Set

$$w_n = \begin{cases} W_{n+1} - W_n - \mathbf{w}_n & \text{in } B_R \cap \mathbb{R}^d_+, \\ -\mathbf{w}_n & \text{in } B_R \cap \mathbb{R}^d_-, \end{cases}$$

where $\mathbf{w}_n \in H^1(B_R)$ is the unique solution of

$$\operatorname{div}(a\nabla \mathbf{w}_n) + \sigma \mathbf{w}_n = (g_{n+1} - g_n) 1_{B_R \cap \mathbb{R}^d_+} \text{ in } B_R \setminus \mathbb{R}^d_0,$$
$$[a\nabla \mathbf{w}_n \cdot \nu] = h_{n+1} - h_n \text{ on } B_R \cap \mathbb{R}^d_0, \quad \text{and} \quad a\nabla \mathbf{w}_n \cdot \nu - i\mathbf{w}_n = 0 \text{ on } \partial B_R.$$

Here we extend a and σ in B_R by setting $a(x', x_d) = a(x', -x_d)$ and $\sigma(x', x_d) = 0$ for $(x', x_d) \in (\mathbb{R}^{d-1} \times \mathbb{R}_-) \cap B_R$; though we still denote these extensions by a and σ. We also denote 1_Ω the characteristic function of a subset Ω of $\mathbb{R}^d$. We derive from (5.2) and (5.6) that

$$\|\mathbf{w}_n\|_{H^1(B_R)} \le Cm2^{-n/2}. \tag{5.7}$$

In this proof, C denotes a constant independent of n. From the definition of w_n, we have

$$\operatorname{div}(a\nabla w_n) + \sigma w_n = 0 \text{ in } B_R.$$

From (5.6) and (5.7), we derive that

$$\|w_n\|_{H^1(B_R)} \le Cm2^{n/2} \quad \text{and} \quad \|w_n\|_{H^1(\partial B_R \cap \mathbb{R}^d_-)} \le Cm2^{-n/2}. \tag{5.8}$$

Set $S = (0, \cdots, 0, -R/4) \in \mathbb{R}^d$. By [35, Theorem 2] (a three sphere inequality), there exists $r_0 \in (R/4, R/3)$, depending only on R and the Lipschitz and elliptic constants of a such that

$$\|w_n(\cdot - S)\|_{\mathbf{H}(\partial B_{r_0})} \le C \|w_n(\cdot - S)\|^{2/3}_{\mathbf{H}(\partial B_{R/4})} \|w_n(\cdot - S)\|^{1/3}_{\mathbf{H}(\partial B_{R/3})},$$

where

$$\|\varphi\|_{\mathbf{H}(\partial B_r)} := \|\varphi\|_{H^{1/2}(\partial B_r)} + \|a\nabla\varphi \cdot \nu\|_{H^{-1/2}(\partial B_r)}.$$

This implies, by (5.8),

$$\|w_n(\cdot - S)\|_{\mathbf{H}(\partial B_{r_0})} \le Cm2^{-n/6}.$$

By Lemma 1, we obtain

$$\|w_n(\cdot - S)\|_{H^1(B_{r_0})} \le Cm2^{-n/6}. \tag{5.9}$$

Since $\mathbf{w}_n$ converges in $H^1(B_R)$ by (5.7), it follows that (W_n) converges in $H^1(B_r \cap \mathbb{R}^d_+)$ with $r := r_0 - R/4$. Let W be the limit of W_n in $H^1(B_r \cap \mathbb{R}^d_+)$. Then

$$\operatorname{div}(a\nabla W) + \sigma W = g \text{ in } B_r \cap \mathbb{R}^d_+, \quad W = 0 \text{ on } B_r \cap \mathbb{R}^d_0, \quad a\nabla W \cdot \eta = 0 \text{ on } B_r \cap \mathbb{R}^d_0.$$

This contradicts the non-existence of W. Hence (5.5) holds. The proof is complete. $\square$

Remark 14. Lemma 10 is inspired by [29, Lemma 2.4]. The proof also has roots from there. The fact that r does not depend on σ is somehow surprising. This is based on a new three spheres inequality in [35, Theorem 2]. Proposition 2 is in the same spirit of the results in [29] and [16] and extends the results obtained there.

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